The 9th International Conference on Differential and Functional Differential Equations

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ABSTRACTS

Moscow, Russia,
June 28 – July 5, 2022

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The 9th International Conference on Differential and Functional Differential Equations

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Lomonosov Moscow State University

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List of 45-minute Invited Lecturers:
O. N. Ageev, A. A. Amosov, A. I. Aptekarev, M. Ben-Artzi, G. A. Bocharov, H. G. Bock, V. M. Buchstaber, G. Crasta, A. A. Davydov, S. Yu. Dobrokhotov, Yu. A. Dubinskii, F. Golse, V. Z. Grines, A. N. Karapetyants, S. B. Kuksin, G. G. Lazareva, K. Lee, A. V. Mikhailov, V. E. Nazaikinskii, A. I. Neishtadt, S. P. Novikov, P. I. Plotnikov, A. I. Shafarevich, I. Shafrir, A. E. Shishkov, A. A. Shkalikov, T. A. Suslina, I. A. Taimanov, D. V. Treschev, V. V. Vlasov, V. A. Volpert, L. Veron, R. Yang.

Krasovskii damping problem for a multidimensional control system of retarded type

A. Sh. Adkhamova
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In [1], N. N. Krasovskii considered the damping problem for a control system with aftereffect described by differential-difference equations of retarded type. He reduced this problem to the boundary-value problem for a system of differential-difference equations with the deviating argument in lower order terms. In [2], Krasovskii's problem was generalized to the case where a control system is described by an equation of neutral type. In [3], a model with constant matrix coefficients and several delays was considered, while [4] was devoted to that one with variable matrix coefficients and several delays.

We consider the problem of bringing a linear non-stationary control system with delay to an equilibrium state of, where the system is described by differential-difference equations of retarded type with variable matrix coefficients and several delays.

The relationship between the variational problem for a nonlocal functional describing the multidimensional control system with delays and the corresponding boundary-value problem for the system of differential-difference equations is established. The existence, uniqueness, and smoothness of a generalized solution to this boundary-value problem are proved.

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On realizations of dynamical systems

O. N. Ageev
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Let T be a measure preserving dynamical system of the Lebesque space. The set of all selfjoinings of T, i.e. the $T \times T$-invariant measures with the standard marginals, is one of the natural dynamical invariants of T. While the joinings theory became nowadays a powerful tool to study dynamical systems, the $T \times T$ equipped with a joining is not so well understood on its own. One of the first intriguing questions is here to reveal what kind of general dynamical systems can be realized in this form.

We claim that, for a typical $T \times T$-invariant measure, $T \times T$ is isomorphic to T for all but a meager subset of dynamical systems T. This means that almost all joinings are graphs. Applying mostly spectral invariants arguments, we try to calculate the difference between T and $T \times T$ in the case of joinings sitting on finitely many graphs.

On the solvability of radiative-conductive heat transfer problems in systems of opaque and semitransparent for radiation bodies

A. A. Amosov
National Research University "Moscow Power Engineering Institute," Moscow, Russia

We give a review of the latest results on the unique solvability of stationary and nonstationary radiative-conductive heat transfer problems in systems of opaque and semitransparent for radiation bodies. Some of these results are published in [1–4].

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Multiple orthogonal polynomials with respect to Hermite weights: applications and asymptotics

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The talk is based on the joint work [1] with S. Yu. Dobrokhotov, A. V. Tsvetkova (Ishlinsky Institute for Problems in Mechanics RAS), and D. N. Tulyakov (Keldysh Institute of Applied Mathematics RAS).

We start with the definition of the Hermite multiple orthogonal polynomials by means of orthogonality relations. Then we present several recent applications, such as eigenvalues distribution of random matrices ensembles with external field and Brownian bridges. The main goal of the talk will be the asymptotics of this polynomial

sequence when the degree of the polynomial is growing in the scale corresponding to its variable (so-called Plancherel–Rotach type asymptotics). The starting point for our asymptotical analysis is the recurrence relations for multiple orthogonal polynomials. We will present an approach based on the construction of decompositions of bases of homogeneous difference equations. Another approach, based on the semiclassical asymptotics in the case of complex-valued phases, will be presented in S. Yu. Dobrokhotov's talk.

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A posteriori estimates for obstacle problems

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Obstacle problems for elliptic and parabolic equations arising in various branches of science and technology are well studied from the mathematical point of view. These studies are mainly focused either on the existence of the unique minimizer or on regularity properties of minimizers and respective free boundaries. A systematic overview of these results can be found in [1, 2].

In this talk, we will touch on another issue. We study the guaranteed bounds of the difference between the exact solution (*minimizer*) of the corresponding variational problem and any function (*approximation*) from the energy class satisfying the prescribed boundary conditions and the restrictions stipulated by the obstacle. They can be called *estimates of deviations* from the exact solution (or a posteriori estimates of functional type). The estimates bound a certain measure (norm) of the error by a functional (error majorant) that depends on the problem data and approximation type, but do not explicitly depend on the exact solution. Hence the functional is fully computable and can be used to evaluate the accuracy of an approximation. Within the framework of this conception, the estimates should be derived on the functional level by the same tools as commonly used in the theory of partial differential equations. They do not use specific features of approximations (e.g., the Galerkin orthogonality) typical for a posteriori methods applied in mesh adaptive computations based upon finite element technologies. Unlike the a priori rate convergence estimates that establish general asymptotic properties of an approximation method, these a posteriori estimates are applied to a particular solution and allow us to directly verify its accuracy.

We discuss such type estimates for the parabolic obstacle problem [3], for elliptic and parabolic thin obstacle problems [3, 4], as well as for elliptic biharmonic obstacle problem [5]. The obtained estimates only depend on the approximate solution (which is known) and on the data of the problem. We emphasise that they also do not need knowledge on the exact coincidence set associated with the exact solution. The obtained error majorants are non-negative and vanishes if and only if the approximation coincides with the exact minimizer.

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Local regularity of weak solutions to a class of strongly nonlinear parabolic systems

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We study regularity of weak solutions of quasilinear elliptic and parabolic systems with nondiagonal principal matrices and quadratic (strong) nonlinearities in the gradient of additional terms.

Partial regularity problem for such systems is usually studied in the class of *bounded* weak solutions under some smallness assumption for their L^∞-norms.

In the case of *scalar* equations with quadratic nonlinearities in the gradient, regularity problem was also studied in the class of *bounded* weak solutions. As is known, the so-called "one-side condition" on the nonlinear term is enough to obtain a priori estimate of the L^∞-norm of a solution to the Dirichlet (the Cauchy-Dirichlet) problem. One can also obtain further regularity of the solution provided that all data of the problem are smooth enough.

It appeared that the one-side condition guarantees boundedness but not further smoothness of weak solutions of such boundary-value problems for the *diagonal* systems. It means that a singular set can appear in such situation. Certainly, weak solutions of the *nondiagonal* systems also admit singularities. Moreover, the maximum principal does not hold for such systems, and one can not obtain a priori estimate of the L^∞ norm of weak solutions. The author started to study regularity of weak (may be unbounded) solutions for nondiagonal systems with strongly nonlinear terms under the one-side condition in [1–3]. Now we modified the proof and relaxed smoothness assumption for the principal matrix up to the optimal one to study local smoothness of weak solutions.

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Antisymmetrized Gelfand–Kapranov–Zelevinskij systems

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Consider the Lie group $GL_n(\mathbb{C})$ and the space of functions on this group. The simplest example are matrix elements a_i^j occuring at the intersection of ith row and jth column. Using these function one can compose other functions, for example, the following minors: $a_{i_1,...,i_k} = det(a_i^j)_{i=i_1,...,i_k}^{j=1,...,k}$.

These function satisfy some relations (the so-called Plucker relations). Polynomials in $a_{i_1,...,i_k}$ that vanish under these relations form an ideal that we denote as Pl. In the problems of the representation theory it is necessary to prove some equalities for polynomials in $a_{i_1,...,i_k}$ that hold molulo Pl:

$$f_1(a) + ... + f_m(a) = 0 \, mod \, Pl.$$

To do it, we chose the following strategy. We define the antisymmetrized Gelfand–Kapranov–Zelevinskij system (A-GKZ for short) as the following system of PDEs for a function $F(a)$ of $a_{i_1,...,i_k}$:

$$\forall g(a) \in Pl, \quad g(\frac{d}{da})F(a) = 0.$$

Then

$$f_1(a) + ... + f_m(a) = 0 \, mod \, Pl \;\Leftrightarrow\; f_1(\frac{d}{da}) + ... + f_m(\frac{d}{da})F = 0$$

for all solution of the A-GKZ system. Since one can only take the generators of the ideal in the definition of the A-GKZ system, it turns out that the A-GKZ system can be written explicitely as follows. For all possible indices $i < j < y$ and subsets $Y \subset \{1, ..., n\}$ consider the following equations:

$$\frac{\partial^2 F}{\partial a_{1,...,i-i,i,Y} \partial a_{1,...,i-i,j,y,Y}} - \frac{\partial^2 F}{\partial a_{1,...,i-i,j,Y} \partial a_{1,...,i-i,i,y,Y}} + \\ + \frac{\partial^2 F}{\partial a_{1,...,i-i,y,Y} \partial a_{1,...,i-i,i,j,Y}} = 0.$$

Removing the last term, we arrive at the system

$$\frac{\partial^2 F}{\partial a_{1,\dots,i-i,i,Y}\partial a_{1,\dots,i-i,j,y,Y}} - \frac{\partial^2 F}{\partial a_{1,\dots,i-i,j,Y}\partial a_{1,\dots,i-i,i,y,Y}} = 0,$$

which is an example of the original Gelfand–Kaparnov–Zelevinsly hypergeometric system. We explain in the talk how one can explicitely construct the solution space of the A-GKZ system using solutions of the GKZ system, and also how this construction helps to solve the original problems in the representation theory.

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Solvability of an operator Riccati integral equation in a reflexive Banach space

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If $X_{1,2}$ are Banach spaces, then by $\mathcal{L}(X_1, X_2)$ we denote the space of all bounded operators acting from X_1 to X_2. By $C_s(\mathcal{I}; \mathcal{L}(X_1, X_2))$ we denote the space of all strongly continuous operator functions on the segment $\mathcal{I} = [0, T]$ with topology of the strong uniform convergence.

An operator function $\{U_{t,s}\}_{0\leqslant s\leqslant t\leqslant T}$ on a Banach space is called *forward* (*in time*) *evolution family* if $U_{t,t} = I$ and $U_{t,s} = U_{t,r}U_{r,s}$ for all $0 \leqslant s \leqslant r \leqslant t \leqslant T$. An operator function $\{V_{s,t}\}_{0\leqslant s\leqslant t\leqslant T}$ on a Banach space is called *backward* (*in time*) *evolution family* if $V_{t,t} = I$ and $V_{s,t} = V_{s,r}V_{r,t}$ for all $0 \leqslant s \leqslant r \leqslant t \leqslant T$. An evolution family is called *strongly continuous* if it is strongly continuous in t (for s fixed) and in s (for t fixed).

Let X be a reflexive Banach space with duality pairing $\langle f, x\rangle$ ($x \in X, f \in X^*$). If $A_1 \in \mathcal{L}(X, X^*)$, then, taking into account the canonical isomorphism between X and X^{**}, one can consider the adjoint operator A_1^* as an element of $\mathcal{L}(X, X^*)$. An operator $A_1 \in \mathcal{L}(X, X^*)$ is self-adjoint if $A_1 = A_1^*$. A self-adjoint operator $A_1 \in \mathcal{L}(X, X^*)$ is non-negative if $\langle A_1x, x\rangle \geqslant 0$ for all $x \in X$.

Analogously, if $A_2 \in \mathcal{L}(X^*, X)$, then we have $A_2^* \in \mathcal{L}(X^*, X)$. An operator $A_2 \in \mathcal{L}(X^*, X)$ is self-adjoint if $A_2 = A_2^*$. A self-adjoint operator $A_2 \in \mathcal{L}(X^*, X)$ is non-negative if $\langle x, A_2x\rangle \geqslant 0$ for all $x \in X$.

Note that if $U_{t,s}$ is a strongly continuous evolution family in a reflexive space X, then $V_{s,t} = U_{t,s}^*$ is a strongly continuous backward evolution family in X^*.

Theorem. *Let X be a reflexive Banach space and the following assumptions hold:*

1. *$\{U_{t,s}\}_{0\leqslant s\leqslant t\leqslant T}$ is a strongly continuous and uniformly bounded forward evolution family in $\mathcal{L}(X)$. Then $V_{s,t} = U_{t,s}^*$ is also a strongly continuous and uniformly bounded backward evolution family in $\mathcal{L}(X^*)$;*

2. *$C \in C_s(\mathcal{I}; \mathcal{L}(X, X^*))$ and $B \in C_s(\mathcal{I}; \mathcal{L}(X^*, X))$;*

3. *$C(t) = C^*(t) \geqslant 0$ and $B(t) = B^*(t) \geqslant 0$ for all $t \in \mathcal{I}$.*

Then for all self-adjoint non-negative operators $G \in \mathcal{L}(X, X^*)$ the (backward) integral Riccati equation

$$P(t) = V_{t,T} G U_{T,t} + \int_t^T V_{t,s}\{C(s) - P(s)B(s)P(s)\}U_{s,t}ds$$

has a unique self-adjoint non-negative solution $P \in C_s(\mathcal{I}; \mathcal{L}(X, X^*))$.

Some applications to the system of linear forward-backward evolution equations

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t) \\ -C(t) & -A^*(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{matrix} x(0) = x_0 \\ y(T) = Gx(T) \end{matrix} \quad t \in [0, T]$$

and the mean-field game system of PDEs will be given.

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The second order of accuracy difference schemes for integral type time-nonlocal parabolic problems

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In paper [1], unique solvability of the nonlocal problem

$$\begin{cases} \frac{du}{dt} + Au = f(t), \ 0 < t < T, \\ u(0) = \int\limits_0^T \alpha(s)Bu(s)ds + \varphi \end{cases} \tag{1}$$

for a parabolic equation in a Hilbert space H with self-adjoint positive definite operators A and B is studied.

In paper [2], the first order accuracy difference scheme and the Cranck–Nicolson difference scheme for approximate solution of problem (1) are investigated.

In this talk, the second order accuracy difference schemes for approximate solution of the problem are studied. Theorems on stability of the two difference schemes, namely, the r-modified Cranck–Nicolson difference scheme and the second order accuracy difference scheme involving the A^2 term, are established. (see [3]). Numerical calculations for both difference schemes are carried out and illustrations with computer results are demonstrated.

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Second-order ntegro-differential equation with difference kernels and inhomogeneity in the linear part

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In connection with applications to the theories of filtration, shock waves, heat exchange, and others (see [1–3]), the solvability of the nonlinear integro-differential equation

$$u^{\alpha}(x) = \int_0^x h(x-t)u'(t)dt + \int_0^x k(x-t)u''(t)dt + f(x), \quad x > 0, \quad \alpha > 1, \tag{1}$$

is considered.

We look for solutions of equation (1) in the class

$$Q_0^2 = \left\{u(x) : \ u \in C^2(0,\infty), \ u(0) = u'(0) = 0 \text{ and } u(x) > 0 \text{ for } x > 0\right\}.$$

Throughout what follows, the kernels $h \in C^2[0,\infty)$, $k \in C^3[0,\infty)$ and the inhomogeneity $f \in C^2[0,\infty)$ are assumed to satisfy the following conditions:

1. $h''(x)$ nondecreasing on $[0,\infty)$, $h(0) = h'(0) = 0$, and $h''(0) \geqslant 0$;

2. $k'''(x)$ nondecreasing on $[0,\infty)$, $k(0) = k'(0) = k''(0) = 0$, and $k'''(0) > 0$;

3. $f(x)$ nondecreasing on $[0,\infty)$, $f(0) = f'(0) = 0$, and $\sup\limits_{0<x\leqslant b} \dfrac{f^{(\alpha-1)/\alpha}(x)}{x^2} < \infty$.

Since equations of form (1) are explicitly solvable only in few particular cases, we see that both for the theory and applications, the development of approximate methods for equations of this type is of great importance.

The report will present new a priori bounds for solutions of equation (1). Using these bounds enables us to prove by weighted metrics method the existence and uniqueness of a solution of integro-differential equation (1) and to define the precise boundaries of the solution. We show that the solution may be obtained by the method of sequential approximations for which the error bound and that of the degree of

their convergence to the precise solution are given. In the case of Lebesgue spaces, convolution-type equations with monotone nonlinearity were studied in [4].

This work was supported by the State contract of the Russian Ministry of Education and Science (contract FEGS-2020-0001).

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Smooth solution of the second initial-boundary value problem for a parabolic system in a nonsmooth domain on the plane

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We consider the second initial-boundary value problem for one-dimensional (with respect to the spatial variable) second-order Petrovsky parabolic systems with variable coefficients in a semibounded domain Ω with nonsmooth lateral boundary which admits, in particular, cusps. A solution to the problem in the class $\underset{0}{C}^{2,1}(\overline{\Omega})$ is constructed by applying the boundary integral equation method. This solution has the form of a special parabolic potential.

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Analysis and numerics for (non)local models and differential forms with Lavrentiev gap: beyond regularity

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Lavrentiev gap is the key phenomenon in calculus of variations to study many properties of problems with general growth. It leads to many challenges in analysis and numerics such as non-density of smooth functions and disconvergence of numerical methods. We present new density results and examples on Lavrentiev gap for general classes of differential forms and non-local models using fractal contact sets. We also construct examples for the general classes of non-local problems and design the finite element scheme to study numerically so-called W-minimizers for such kind of problems.

This talk is based on several joint works with Lars Diening, Moritz Kassmann (Bielefeld), Mikhail Surnachev (Keldysh Institute of Applied Mathematics, Moscow), Johannes Storn (Bielefeld) and Christoph Ortner (UBC, Vancouver).

Stationary spherically symmetric solutions of the Vlasov–Poisson System depending on local energy

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We consider the Vlasov–Poisson system of equations in the three-dimensional case, modelling distribution of gravitating matter in stellar dynamics. A stationary spherically symmetric solution of this system is a triple (f, ρ, U) of the following functions: the distribution function $f = f(r, u)$, the local density $\rho = \rho(r)$, and the Newtonian potential $U = U(r)$, where $r = |x|$, $u = |v|$, $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ are the space–velocity coordinates. In this lecture we consider the following problem: for a given function $p = p(r)$ we obtain sufficient conditions for p to be "extendable," which means that there exists a stationary spherically symmetric solution (f, ρ, U) of the Vlasov–Poisson system with f depending on the local energy $E := U(r) + \dfrac{u^2}{2}$ such that $\rho = p$, see [1]. A proof is based on the reduction of this problem to Eddington's equation.

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Probabilistic representation of a solution to the Cauchy–Neumann problem for a nonlinear parabolic equation

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Consider the Cauchy–Neumann problem for a nonlinear parabolic equation

$$\frac{\partial v(t,y)}{\partial t} = \frac{1}{2} Tr[\nabla^2 (AA^*)(y, [\rho * v](t,y)) v(t,y)] + c(y, [\rho * v](t,y)) v(t,y), \qquad (1)$$

$$v(0,y) = v_0(y), \quad div(AA^*)(y, [\rho * v])](t,y)) \cdot n = 0, \quad y \in \partial G.$$

in the half-space $G = \mathbb{R}^d_+$ with boundary $\partial G = \{x_d = 0\}$ and an outer normal vector $n(x)$. Hereρ is a mollifier, $[\rho * v](t,y) = \int_{\mathbb{R}^d_+} \rho(y-x) v(t,x) dx$, and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Consider the stochastic system

$$d\xi(t) = A(\xi(t), [\rho * v(t, \zeta(t)]) dw(t) - \chi_{\partial G}(\xi(t)) n(\xi(t)) dk(t), \qquad (2)$$

$$\xi(0) = \xi_0, \quad k(0) = 0,$$

$$v(t,y) = E\left[\rho(y - \xi(t)) exp\left\{\int_0^t c(\xi(s), [\rho * v](s, \xi(s)) ds\right\}\right], \qquad (3)$$

where $\chi_{\partial G}(x)$ is the characteristic function of ∂G, $w(t) \in \mathbb{R}^d$ is a Wiener process, a random variable ξ_0 does not depend on $w(t)$, $P(\xi_0 \in dy) = v_0(y)dy$, and $k(t)$ is the local time.

Theorem 1. *Let $A(x,v) \in \mathbb{R}^d \otimes \mathbb{R}^d, c(x,v) \in \mathbb{R}$ be Lipschitz continuous and bounded. Then there exists a solution $(\xi(t), k(t), v(t,y))$ of (2) and (3), and the function v defined by (3) is a weak solution of (1).*

To prove this assertion, we apply an approach developed in [1]. Namely, we construct a special extension of (2), (3) to a stochastic system in $\mathbb{R}^d$, using a map $\Gamma : C([0,T]; \mathbb{R}^d) \to C([0,T]; \mathbb{R}^d_+)$, $\Gamma(\zeta) = \xi$, called the Skorokhod map. In $\mathbb{R}^d$ we derive an alternative stochastic system giving rise to a stochastic process $\tilde{\xi}(t)$ such that characteristic functions of $\xi(t)$ and $\tilde{\xi}(t)$ are equivalent (see [2]). This allows us to construct a probabilistic representation of a weak solution of the Cauchy problem for the extended PDE. As a consequence, we can verify that a weak solution to (1) is constructed in this way.

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On some examples of stationary solutions to the Vlasov–Poisson equations in a finite cylinder

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The Vlasov–Poisson equations in domains with a boundary describe the kinetics of charged particles of high-temperature plasma in controlled thermonuclear fusion devices. Mixed problems for the Vlasov–Poisson system for a two-component plasma in cylindrical domains in particular describe the kinetics of charged particles of high-temperature plasma in a "mirror trap." We consider the Vlasov–Poisson system for two-component high-temperature plasma in a finite cylinder. We construct the stationary solutions corresponding to a plasma confined in a "mirror trap." The supports touch the boundary of the domain only in two small prescribed discs at the top and the bottom of the cylinder.

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From crystals to Dirac operators spectral theory

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We consider constant coefficient first-order partial differential systems, homogeneous and non-homogeneous, and their potential perturbations. It is assumed that the homogeneous part is strongly propagative. In the non-homegeneous case it is assumed that the operator is isotropic. The spectral theory of such systems and their potential perturbations is expounded, and a Limiting Absorption Principle is

obtained up to thresholds. Special attention is given to a detailed study of the Dirac (massive and massless) and Maxwell operators.

The estimates of the spectral derivative near the thresholds are based on trace estimates on the slowness surfaces. Two applications of these estimates are presented:

1) Global spacetime estimates of the associated evolution unitary groups, commonly viewed as decay estimates. While analogous estimates exist for the Dirac operator, our decay estimates for the Maxwell system are completely new.

2) The finiteness of the eigenvalues (in the spectral gap) of the perturbed Dirac operator is studied under suitable decay assumptions on the potential perturbation.

This is joint work with T. Umeda (Japan).

Multiple hypergeometric functions and applications

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Hypergeometric functions in several variables $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ considered in the talk are defined with the help of the following N-multiple series [1]:

$$\Phi^{(N)}(L, \mathbf{g}, \mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{j=1}^{M} \frac{\Gamma(\gamma_j)}{\Gamma(\sum_{s=1}^{N} \lambda_{s,j} k_s + \gamma_j)} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}!}. \tag{1}$$

Here $\mathbf{k} = (k_1, \ldots, k_N)$ is the multi-index, $k_s \in \mathbb{Z}^+$, $|\mathbf{k}| := \sum_{s=1}^{N} k_s$, $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} \cdots z_N^{k_N}$, $\mathbf{k}! := k_1! \cdots k_N!$; $L = \{\lambda_{s,j}\}$ is an integer $(N \times M)$-matrix whose elements satisfy the relations $\sum_{j=1}^{M} \lambda_{s,j} = -1$, $s = \overline{1, N}$; $\mathbf{g} = (\gamma_1, \ldots, \gamma_M) \in \mathbb{C}^M$ is a vector parameter, and $\Gamma(x)$ is the gamma function. The series in formula (1) belong to the Horn class of hypergeometric series, see [1, 2].

The research presents an approach to obtaining formulas for the analytical continuation of series (1) by variables $\mathbf{z}$ to the whole complex space $\mathbb{C}^N$ in the form of linear combinations $\Phi^{(N)}(\mathbf{z}) = \sum_m A_m u_m(\mathbf{z})$, where $u_m(\mathbf{z})$ are hypergeometric series of the Horn type satisfying the same system of partial differential equations as series (1), and A_m are some coefficients. The implementation of this approach is demonstrated by the example of the Lauricella hypergeometric function $F_D^{(N)}$. This function is defined in the unit polydisk $\mathbb{U}^N := \left\{|z_j| < 1,\ j = \overline{1, N}\right\}$ by the following series [3]:

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1} \cdots (a_N)_{k_N}}{(c)_{|\mathbf{k}|} k_1! \cdots k_N!} \mathbf{z}^{\mathbf{k}}. \tag{2}$$

Here the complex values $(a_1, \ldots, a_N) =: \mathbf{a}$, b, and c play the role of parameters, $c \notin \mathbb{Z}^-$, and the Pochhammer symbol is defined as $(a)_m := \Gamma(a + m)/\Gamma(a)$. A complete set of formulas for the analytical continuation of series (2) for an arbitrary N is constructed in [4].

We then apply the obtained results on the analytic continuation of the Lauricella function $F_D^{(N)}$ to the effective solution of the parameter problem for the Schwarz–Christoffel integral in the crowding situation, and to the computation of conformal maps of polygonal domains of complex shape.

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Shadowing property re(al)visited

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When modeling a time-evolving process, we obtain its approximate realizations. This proximity is due to several reasons. First, we never exactly know the description of the process itself, and second, the presence of various kinds of errors from purely random to rounding errors when implemented on a computer are inevitable. The question of the adequacy of the simulation results is primarily associated with the presence of a real trajectory of the process under study in the vicinity of the obtained realization over the longest possible time interval. This question is especially nontrivial in the case of a chaotic system, since for such systems close trajectories diverge very quickly (often exponentially fast).

At the level of connections between individual trajectories of a hyperbolic system and the corresponding pseudo-trajectories[1], this problem was first posed by D. V. Anosov [1] as a key step of the analysis of structural stability of diffeomorphisms. A similar but much less intuitive approach called "specification" in the same setting was proposed by R. Bowen [2]. Informally, both approaches ensure that errors do not accumulate during the process of modeling: in systems with the shadowing property, each approximate trajectory can be uniformly traced by a true one for an arbitrary long period of time. Naturally, this is of great importance in chaotic systems, where even an arbitrary small error in the starting position lead to (exponentially in time) large divergence of trajectories.

Further development demonstrated that for a diffeomorphism the shadowing property implies the uniform hyperbolicity. To some extent, this limits the theory of uniform shadowing to an important but very special class of hyperbolic dynamical

[1] Approximate trajectories of a system under small perturbations.

systems. The concept of average shadowing introduced in [3] about 30 years ago gave a possibility to extend significantly the range of perturbations under consideration in the theory of shadowing, in particular to be able to deal with perturbations which are small only on average but not uniformly.

The most notorious in the variety of obstacles in the analysis of the shadowing property is that one needs to take into account an infinite number of independent perturbations of the original system. This makes the problem highly nonlocal. It is therefore very desirable to reduce the shadowing problem to the situation with a single perturbation, albeit with tighter control of the approximation accuracy.

To realize this idea, we developed in our recent paper [4] a fundamentally new "gluing" construction consisting in the effective approximation of a pair of consecutive segments of true trajectories.

We restrict ourselves to discrete time dynamical systems, leaving the extension of our approach to continuous time systems (flows) for future research. A discrete time dynamical system is completely defined by a map $T : X \to X$, not necessarily invertible, from a metric space (X, ρ) into itself.

A *trajectory* of the map T starting at a point $x \in X$ is a sequence of points $\vec{x} := \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\} \subset X$, for which $x_0 = x$ and $Tx_i = x_{i+1}$ for all available indices i.

A *pseudo-trajectory* of the map T is a sequence of points $\vec{y} := \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\} \subset X$, for which the sequence of distances $\{\rho(Ty_i, y_{i+1})\}$ for all available indices i satisfies a certain "smallness" condition.

Given $\varepsilon > 0$, we say that a pseudo-trajectory $\vec{y}$ is of

(U) *uniform* type if $\rho(Ty_i, y_{i+1}) \leqslant \varepsilon$ for all available indices i.

(A) *small on average* type if $\limsup\limits_{n\to\infty} \frac{1}{2n+1} \sum\limits_{i=-n}^{n} \rho(Ty_i, y_{i+1}) \leqslant \varepsilon$.

The idea of *shadowing* in the dynamical systems theory boils down to the following question: is it possible to approximate pseudo-trajectories of a given dynamical system by true trajectories? Naturally, the answer depends on the type of the approximation.

We say that a true trajectory $\vec{x}$ *shadows* a pseudo-trajectory $\vec{y}$ with accuracy δ (notation δ-shadows):

(U) *uniformly* if $\rho(x_i, y_i) \leqslant \delta$ for all available indices i.

(A) *on average* if $\limsup\limits_{n\to\infty} \frac{1}{2n+1} \sum\limits_{i=-n}^{n} \rho(x_i, y_i) \leqslant \delta$.

We say that a DS (T, X, ρ) satisfies the $(\alpha + \beta)$-*shadowing property* (notation $T \in \mathcal{S}(\alpha, \beta)$) with $\alpha \in \{U, A, A', R\}$, $\beta \in \{U, A\}$ if $\forall \delta > 0\ \exists \varepsilon > 0$ such that each ε-pseudo-trajectory of α-type can be shadowed in the β sense with the corresponding accuracy δ.

We say that a trajectory $\vec{z}$ *glues* together semi-trajectories $\vec{x}, \vec{y}$ with accuracy rate $\varphi : \mathbb{Z} \to \mathbb{R}_+$ *strongly* if

$$\rho(x_k, z_k) \leqslant \varphi(k)\rho(x_0, y_0) \ \ \forall k < 0, \quad \rho(y_k, z_k) \leqslant \varphi(k)\rho(x_0, y_0) \ \ \forall k \geqslant 0.$$

In other words, $\vec{z}$ approximates both the backward part of $\vec{x}$ and the forward part of $\vec{y}$ with accuracy controlled by the rate function φ.

We say that the DS (T, X, ρ) satisfies the *gluing property* with the rate-function $\varphi : \mathbb{Z} \to \mathbb{R}$ (notation $T \in G(\varphi)$) if for any pair of trajectories $\vec{x}, \vec{y}$ there is a trajectory $\vec{z}$ gluing them at time $t = 0$ with accuracy φ in the strong/weak sense.

Our main result is the following statement.

Theorem 1. *Let $T : X \to X$ be a map from a metric space (X, ρ) into itself, and let $T \in G(\varphi)$ with $\sum_k \varphi(k) < \infty$. Then $T \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$.*

Now we are going to demonstrate our approach for some important classes of dynamical systems (in particular, for non-invertible and discontinuous ones).

Example 1 (Affine mapping)**.** Let $X := \mathbb{R}^d$ with $d \geqslant 1$ with the euclidean metric ρ, A be a $d \times d$ matrix, and $a \in \mathbb{R}^d$. Then $Tx := Ax + a \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$ if and only if $E^n = \varnothing$ and either $E^s = \varnothing$ or $E^u = \varnothing$.

Here E^0, E^s, E^u, E^n are linear subspaces of $\mathbb{R}^d$, corresponding to the zero eigenvalue of A, the remaining contracting part of A, the expanding part of A, and the neutral part (corresponding to the eigenvalues of modulus 1).

Example 2 (Anosov diffeomorphism)**.** Let $X := \mathbb{T}^2$ be a unit 2-dimensional torus and let $T : X \to X$ be a uniformly hyperbolic diffeomorphism. Then $T \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$.

Example 3 (Nonuniform hyperbolicity)**.** We set $X := [0, 1]$, $\alpha, \beta \geqslant 0$, $0 < c < 1$, and

$$Tx := \begin{cases} x(1 + ax^\alpha) & \text{if } x \leqslant c, \\ 1 - (1 - x)(1 + b(1 - x)^\beta) & \text{if } x > c. \end{cases}$$

Then $T \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$ iff $0 \leqslant \alpha, \beta < 1$ and $c(1 + ac^\alpha) = (1 - c)(1 + b)^\beta = 1$.

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Mathematics for immunology

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Immune system functions to protect the host organisms against life-threatening infections. Modern research in immunology is characterized by a tremendous rise of data characterizing its functioning on multiple levels of detail. Mathematics provides the analytical tools for representing the immune system structure, identifying the regulation principles and predicting the immune response to perturbations of the host antigenic homeostasis [1–3]. In addition to traditional approaches based on representing the population dynamics of cells and molecules with systems of ordinary-, functional-, or partial differential equations, the emerging integrative analysis considers the methods of graph theory, stochastic processes, and artificial neural networks. We discuss a number of recent applications of various mathematical tools in immunology. These include

(i) the graph-type model of the human lymphatic system;

(ii) computational geometry methods to model the lymph node structure at high spatial resolution;

(iii) neural network-based approximation of the lymph node drainage function;

(iv) Markov chain Monte Carlo-based stochastic modelling of virus life cycle (HIV-1, SARS-CoV-2) to identify the target for therapy.

Finally, some theoretical foundations of immunology are discussed, e.g., the "balance of growth and differentiation" concept of the immune system response to antigenic perturbation as proposed by Grossman and Paul [4]. Modelling arguments are presented that it can be consistently linked to the Sobolev norm of the cumulative viral load function.

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Inverse optimal control problems and application to modeling the gait of cerebral palsy patients

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We present numerical methods solving inverse optimal control problems as complex bi-level dynamic optimization problems: a nonlinear approximation problem on the upper level and a nonlinear optimal control problem (OCP) with discontinuities and mixed path-control constraints on the lower level. The OCP solution can be considered as a model that describes autonomous optimal processes in nature such as human gait. However, the optimal control model includes unknown parameters that need to be determined by fitting its solution to measurements in the upper level optimization. We develop a direct mathematical all-at-once approach for solving this new class of problems and apply this to derive biomechanical optimal control models for the gait of cerebral palsy patients from real-world motion capture data obtained by the Motion Lab of the Orthopedic University Hospital Heidelberg.

Hyperbolization for nonlinear Schrödinger type equations

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Some nonlinear Schrödinger-type equations are considered. They describe various processes in optics, plasma physics, hydrodynamics, biophysics, etc. Sometimes these problems require large time intervals or small time steps in the numerical implementation. To avoid these difficulties, a hyperbolization procedure is applied. Adding the second derivative of an unknown function in time with a special small parameter allows us to create three-layer explicit numerical schemes possessing better stability in comparison with the classical ones. In its turn, this fact allows us to increase a time step. A further advantage is obtained by using an additional damping term connected with the above-mentioned small parameter. The additional terms act as regularization procedures smoothing out non-physical numerical effects.

Two ways of implementation of the hyperbolization procedure are suggested. One of them is based on the spectral operator replacements, where an approximate solution

is obtained by means of the consecutive operator substitutions in the small intervals. In fact, this algorithm is the splitting procedure on physical processes. The proposed algorithm provides continuity of a solution and its derivative in time. The other one employs the conservative unknown functions (field and flow). In the non-divergent case, it is necessary to introduce the third unknown function (the nonlinearity term).

In the case of an unbounded operator, it is necessary first to make transition to a bounded operator or to a semibounded one (which is possible).

The most interesting question is how close the "hyperbolized" solution is to the original one. The damping term improves this problem essentially.

On index of elliptic boundary-value problems associated with isometric group actions

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Let M be a smooth compact manifold with boundary X. We suppose that M is endowed with a Riemannian metric. In a neighborhood of the boundary we use local coordinates $x = (x', x_n)$ on M, where $\dim M = n$, $x' = (x_1, \ldots, x_{n-1})$ are local coordinates on X, and the boundary is locally defined by the equation $x_n = 0$, while M is defined by the inequality $x_n \geqslant 0$. We fix a Riemannian metric on M.

Let us consider Boutet de Monvel operators of zero order and type. We write such operators as follows:

$$\mathcal{D} = \begin{pmatrix} A+G & C \\ B & A_X \end{pmatrix} : \begin{matrix} L^2(M) \\ \oplus \\ L^2(X) \end{matrix} \longrightarrow \begin{matrix} L^2(M) \\ \oplus \\ L^2(X) \end{matrix}, \tag{1}$$

where

- A is a classical zero-order pseudodifferential operator (ψDO) on M whose complete symbol satisfies the transmission property;
- A_X is a zero-order ψDO on X;
- B, C, and G are boundary, coboundary, and Green operators, respectively (or trace, potential, and singular Green operators in the terminology of Boutet de Monvel [1]).

Let us denote the algebra of matrices (1) by $\Psi_B(M) \subset \mathcal{B}(L^2(M) \oplus L^2(X))$ and let Γ be a discrete finitely generated group of isometries $\gamma\colon M \to M$ preserving the boundary, $\gamma(X) = X$.

Given $\gamma \in \Gamma$, we define the shift operator

$$T_\gamma : L^2(M) \oplus L^2(X) \longrightarrow L^2(M) \oplus L^2(X), \quad (u(x), v(x')) \longmapsto (u(\gamma^{-1}(x)), v(\gamma^{-1}(x'))).$$

Elements $\{\mathcal{D}_\gamma\}_{\gamma\in\Gamma}$ in the smooth crossed product (see [2]) $\Psi_B(M) \rtimes \Gamma$ define the operators

$$\{\mathcal{D}_\gamma\}_{\gamma\in\Gamma} \quad \longmapsto \quad \sum_{\gamma\in\Gamma} \mathcal{D}_\gamma T_\gamma : L^2(M) \oplus L^2(X) \to L^2(M) \oplus L^2(X). \tag{2}$$

Operators in (2) are called Γ-*Boutet de Monvel operators*.

We define the conditions under which Γ-Boutet de Monvel operators are elliptic and give the index formula for elliptic Γ-Boutet de Monvel operators.

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Weak superposition principle for signed measure-valued solutions of the continuity equation

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Given a bounded Borel vector field b (possibly non-autonomous), we consider the associated continuity equation for a time-dependent family of signed Borel measures $\{\mu_t\}_t$. It is well known that if these measures are non-negative, then they can be represented as a superposition of Dirac deltas, concentrated along the integral curves of b. This result, known as the *superposition principle*, was extended to signed measure-valued solutions in [1] under additional assumptions on the regularity of b. We explicitly show that without additional assumptions on b the superposition does not hold for generic signed measure-valued solutions, even in the one-dimensional setting. By relaxing the notion of the integral curve we establish a weak version of the superposition principle, which is valid in the one-dimensional setting without any further assumptions on b. We also discuss the relation between the weak superposition principle and uniqueness of the solutions of the initial-value problem.

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On operator estimates for elliptic operators in perforated domains

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We consider the boundary value problem for a second order scalar differential operator H_ε with variable coefficients in a multi-dimensional domain Ω_ε finely perforated by small holes distributed along a given manifold S inside Ω. The size of the holes and the distances between them are controlled by a small parameter ε; the union of the holes is denoted θ^ε. The perforated domain Ω_ε is obtained from Ω by removing the holes θ^ε. The shapes of the holes in θ^ε are arbitrary as well as their distribution along the manifold. The equation we consider in Ω_ε reads as

$$\left(-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} + A_0 - \lambda \right) u_\varepsilon = f$$

for a given complex parameter λ and a given function $f \in L_2(\Omega_\varepsilon)$. On the external boundary of the domain we impose the Dirichlet condition, while the boundaries of small holes are subject either to the Dirichlet condition or to a nonlinear Robin condition. We consider various cases leading to various homogenized problems. These problems are for the same differential equation but subject to some homogenized condition on the manifold S. Our main result provides the estimates for the convergence rates and the main feature is that these estimates are uniform with respect to the function f on the right hand side.

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Integrable polynomial Hamiltonian KdV–Novikov hierarchies and their quantization

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In the first part of the talk, we will discuss finite-dimensional integrable polynomial Hamiltonian hierarchies associated with Novikov's equations for the Korteweg–de Vries hierarchy. The corresponding Poisson brackets and the complete set of polynomial Hamiltonians will be explicitly described.

The well-known construction of the KdV hierarchy is the reduction of the hierarchy on a free associative algebra of an infinite number of variables.

In the second part of the talk, we will give an explicit description of the noncommutative versions of Novikov's equations and their first integrals. The hierarchies on a free associative algebra of $2N$ variables ($N = 1, 2, \ldots$) will be obtained. Using the

examples of $N = 1, 2$, and 3, we will show the general method of quantization ideals introduced recently by A. V. Mikhailov. In our case, we have a two-sided ideal that is invariant with respect to the noncommutative Nth Novikov's equation. The factor over this ideal defines a dynamical system on an associative algebra of $2N$ variables with an additive Poincaré–Birkhoff–Witt basis.

In the third part of the talk, we will describe an invertible polynomial transformation of a free associative algebra of $2N$ variables that allows us to pass from the Poisson bracket in $\mathbb{C}^{2N}$ to commutators. As a result, we obtain N-quantum Novikov's equations and quantum hierarchies in accordance with the Heisenberg approach. The corresponding operator representation of explicitly given quantum Hamiltonians will be presented.

The talk is based on the results obtained jointly with A. V. Mikhailov.

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Weak solutions of boundary value problems for general quasilinear PDEs

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Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain with boundary $\partial\Omega$, $\mathcal{L} = \sum\limits_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha$ be some differential operation with smooth complex $j \times k$-matrix coefficients $a_\alpha(x)$, and $\mathcal{L}^+$ be the formally adjoint differential operation. Let L_0, L_0^+ be the minimal operators. M. Vishik introduced the conditions

$$\textit{the operator } L_0 : D(L_0) \to L_2^j(\Omega) \textit{ has a continuous left-inverse}, \tag{1}$$

$$\textit{the operator } L_0^+ : D(L_0^+) \to L_2^j(\Omega) \textit{ has a continuous left-inverse} \tag{2}$$

and proved that these conditions are necessary and suffucient for the existence of a solvable extension $L_B : D(L_B) \to L_2^j(\Omega)$ (i.e. $D(L_0) \subset D(L_B)$, $\exists L_B^{-1} : L_2^j(\Omega) \to D(L_B)$). L. Hörmander proved the fulfillment of these conditions for any scalar operator with constant coefficients in a bounded domain. For the case of scalar operators

$$L(x, D) = P^0(D) + \sum_i C_i(x)P^i(D),\ C_i \in C^\infty(\overline{\Omega}),\ P^i \in \mathbf{C}[\xi] \tag{3}$$

of the real principal type with ord $P^i \leqslant m - 1, |\nabla P^0(\xi)| \neq 0$ if $\xi \neq 0$, and also for the case of constant strength operators of form (3) with $P^i \prec P^0$ and analytical C^i, conditions (1), (2) follow from results by G. Gudmundsdottir.

We consider the equation

$$\mathcal{L}^+ A\mathcal{L}u = f \tag{4}$$

with some continuous (linear or nonlinear) operator $A : L_2^j(\Omega) \to L_2^j(\Omega)$.

A function $u \in D(L_0)$ satisfying the integral identity $< AL_0\, u, \mathcal{L}v >=< f, v >$ for each function $v \in (C_0^\infty(\Omega))^k$, is called a generalized solution of the Dirichlet problem in Ω for equation (4) with $f \in D'(L_0)$. This is equivalent to the equation $L_0' AL_0 u = f$. The Dirichlet problem is called well-posed if there exists a continuous inverse operator $M : D'(L_0) \to D(L_0)$ to the operator $L_0' AL_0$. Let $P : L_2^j(\Omega) \to \mathrm{Im} L_0$ denote the orthoprojector.

Statement. *The generalized Dirichlet problem for equation* (4) *is well-posed if and only if condition* (1) *is fulfilled and the operator* $PA : Im\, L_0 \to Im\, L_0$ *is a homeomorphism.*

Example. Let $\mathcal{L} = \Box = \partial^2/\partial x_1 \partial x_2$, A be the Nemytsky operator given by $(Au)(x) = \varphi(x, |u|)\, u$, where the function φ is bounded, satisfies the Caratheodory condition and $\varphi(x,t)\, t - \varphi(x,s)s \geqslant m(t-s)$ for $t \geqslant s, m > 0$. Then condition (1) is fulfilled in any bounded domain and the Dirichlet problem for the equation $\Box A \Box u = f$ is well-posed. If the smooth boundary does not contain segments of characteristics, then the relation $u \in D(L_0)$ means $u|_{\partial\Omega} = u_\nu'|_{\partial\Omega} = 0$ almost everywhere on $\partial\Omega$. Instead of $\Box$, one could consider any other scalar operator with property (1) and obtain the same conclusion.

In these considerations we use ideas and arguments of H. Gajewski.

The setting of other boundary value problems are analogous.

Subordination principle and Feynman–Kac formulae for generalized time-fractional evolution equations

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We consider generalized time-fractional evolution equations of the form

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds$$

with a fairly general memory kernel k and an operator L being the generator of a strongly continuous semigroup. In particular, L may be the generator L_0 of a Markov process ξ on some state space Q, or $L := L_0 + b\nabla + V$ for a suitable potential V and drift b, or L generating subordinate semigroups or Schrödinger type groups. This class of evolution equations includes in particular time- and space-fractional heat and Schrödinger type equations.

We show that the subordination principle holds for such evolution equations and obtain Feynman–Kac formulae for solutions of these equations with the use of different stochastic processes, such as subordinate Markov processes and randomely scaled Gaussian processes. In particular, we obtain some Feynman–Kac formulae with generalized grey Brownian motion and other related self-similar processes with stationary increments.

The talk is based on the joint work with Ch. Bender and M. Bormann.

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Controllability of nonlinear third-order dispersion equation

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Third-order dispersion equation is based on the KdV equation. Russel and Zang studied the linear part of the KdV equation. Practically, most systems are nonlinear in nature. We extended the nonlinear KdV equation using very rarely used techniques of the operator theory. Then I dropped the strong condition of the continuity of an operator using the method of integral contractor and regularity. I proved the existence result using the integral contractor of an operator but the uniqueness was not guaranteed which was taken care of by the regularity of the nonlinear operator. Then I proved the controllability of the same nonlinear KdV equation. I would like to extend the same result for the fractional order KdV equation.

On Meyers estimates for Zaremba problem

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Consider a bounded Lipschitz graph plane domain $D = \{(x, y) \ : \ 0 < x < 1, \ 0 < y < 1\}$. We define the Sobolev space of functions $W_2^1(D, F)$, where $F \subset \partial D$ is a Cantor set on the segment $\{(x, y) \ : \ 0 < x < 1, \ y = 0\}$, as a completion of functions infinitely differentiable in the closure of D and equal to zero in a neighborhood of F, with respect to the norm $\| u \|_{W_2^1(D,F)} = \left(\int\limits_D v^2 \, dx + \int\limits_D |\nabla v|^2 \, dx \right)^{1/2}$. We consider the Zaremba problem

$$\Delta u = l \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \tag{1}$$

where $G = \partial D \setminus F$, and $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of the function u, and l is a linear functional on the space $W_2^1(D, F)$.

A variational solution to problem (1) is a function $u \in W_2^1(D, F)$ subject the equality

$$\int\limits_D \nabla u \cdot \nabla \varphi \, dx = \int\limits_D f \cdot \nabla \varphi \, dx$$

for any $\varphi \in W_2^1(D, F)$. Here $f = (f_1, f_2)$, $f_i \in L_2(D)$, appears in the representation of the functional l due to the Hahn–Banach theorem.

Theorem. *If* $f \in \left(L_{2+\delta_0}(D)\right)^n$, *where* $\delta_0 > 0$, *then there are positive constants* $\delta(n, \delta_0) < \delta_0$ *and* C *such that for solution of problem* (1) *the following estimate holds:*

$$\int\limits_D |\nabla u|^{2+\delta} dx \leqslant C \int\limits_D |f|^{2+\delta}\, dx,$$

where C *depends only on* δ_0, *space dimension* n, c_0 *from* (2), *and also constants* L *and* R_0 *involved in the definition of the Lipschitz property of the domain* D.

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Motion of rigid bodies in a viscous fluid with collisions and slippage

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In our presentation we will discuss a paradox related to modeling of the motion of a rigid body in a viscous fluid filling a bounded domain. The motion is described by a system of coupled differential equations: Newton's second law and the Navier-Stokes equations.

The fluid–rigid–body interaction problem has been studied by many authors. These authors assumed a non-slip boundary condition (classical Dirichlet's condition) at the boundaries of the body and of the domain. The assumption led to the very paradoxical result

- *during the motion, there can not be collisions between the body and the domain boundary.*

We will present recent results showing the well-posedness of the fluid–rigid problem when the slippage (Navier's boundary condition) is allowed at the boundaries. This assumption eliminates the non-collision paradox. In the articles [1–3] we showed the well-posedness of the problem, posed in this statement:

- the local-in-time uniqueness was obtained in [1];
- the global-in-time solvability results were proved in [2, 3].

The main two difficulties of the study of the problem are related to:

a) the lack of Sobolev-type embedding theorems [4, 5] in cusp domains: this type of fluid-filled domains appears at the moments of collision between the body and the boundary of the domain;

b) due to the definition of a weak solution, the solutions sought must belong to the BD-bounded deformation space [6].

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On solvability of higher order differential algebraic equations with singular points

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Modeling of various technical and natural processes often results in systems that comprise ordinary differential equations of different orders and algebraic equations (see, for example, [1] and the comprehensive bibliography on the subject therein). In the linear case, such systems can be written in the following form:

$$\Lambda_k x := \sum_{i=0}^{k} A_i(t) x^{(i)} = f, \ t \in T = [\alpha, \beta] \subset \mathbb{R}^1, \tag{1}$$

where $A_i(t)$ are $(n \times n)$-matrices, $x \equiv x(t)$ is a desired vector-function, $f \equiv f(t)$ is a given vector-function, $x^{(i)}(t) = (d/dt)^i x(t), \ x^{(0)}(t) = x(t)$. It is assumed that

$$\det A_k(t) = 0 \ \forall t \in T. \tag{2}$$

We suppose that all entries of (1) are sufficiently smooth for further reasoning and that we are given a set of initial values:

$$\mathrm{x}(\alpha) = \mathrm{a}_x = \begin{pmatrix} a_0^\top & a_1^\top & \dots & a_{k-1}^\top \end{pmatrix}^\top, \tag{3}$$

$\mathrm{x} = \begin{pmatrix} x^\top & \dot{x}^\top & \dots & (x^{(k-1)})^\top \end{pmatrix}^\top$, $a_i \in \mathbb{R}^n$, $i = 1, 2, \dots k-1$.

Systems (1) satisfying (2) are commonly referred to as differential algebraic equations (DAEs) [2].

Generally, singular points of DAEs are not easy to identify. In particular, the points where the rank of the matrix $A_k(t)$ changes are not always singular. The goal of this paper is to formalize the notion of singular points for DAEs, classify them and propose ways for finding them in the domain of a particular DAE. To this end, we employ techniques previously developed for $k = 1$ in [3].

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Oscillation conditions for solutions to first-order delay differential equations

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We say that a real-valued function defined on $\mathbb{R}_+ \equiv [0, \infty)$ *oscillates* if it has a sequence of zeros unbounded from the right.

Unlike solutions to linear ordinary differential equations, solutions to linear differential equations with aftereffect can oscillate.

Consider the equation

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \in \mathbb{R}_+, \tag{1}$$

where $a, h \in C(\mathbb{R}_+)$, $a(t) \geqslant 0$, $h(t) \leqslant t$, $\lim\limits_{t \to +\infty} h(t) = +\infty$. The following well-known theorem generalizes results by Myshkis [1].

Theorem 1 (see [2]). *If* $\varliminf\limits_{t \to +\infty} \int\limits_{h(t)}^{t} a(s)\, ds > 1/e$, *then all solutions of* (1) *oscillate.*

Generalization of Theorem 1 to the equation with several delays

$$\dot{x}(t) + \sum_{k=1}^{n} a_k(t)x(h_k(t)) = 0, \quad t \in \mathbb{R}_+, \tag{2}$$

where $a_k(t) \geqslant 0$, $h_k(t) \leqslant t$, $\lim\limits_{t\to+\infty} h_k(t) = +\infty$, $k = 1, \dots, n$, turned out to be a non-trivial problem. In [3], an example of equation (2) is given where the equation has a non-oscillating solution while $\varliminf\limits_{t\to+\infty} \sum\limits_{k=1}^{n} \int\limits_{h_k(t)}^{t} a_k(s)\,ds > 1/e$.

Define n families of sets $E_k(t) = \{s \geqslant t \mid h_k(s) < t\}$, $t \in \mathbb{R}_+$, $k \in \{1, \dots, n\}$.

Theorem 2 (see [4]). *If* $\varliminf\limits_{t\to+\infty} \sum\limits_{k=1}^{n} \int\limits_{E_k(t)} a_k(s)\,ds > 1/e$, *then all solutions to* (2) *oscillate.*

Note that the scope of Theorem 2 is wider than that of Theorem 1 even in the case $n = 1$.

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Concavity properties of solutions to Robin problems

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Let $\Omega \subset \mathbb{R}^N$ be a uniformly convex domain of class $\mathcal{C}^m$, with $m \geqslant 4 + \frac{N}{2}$.

We call *Robin ground state* of Ω a positive solution to

$$\begin{cases} -\Delta u = \lambda^\beta\, u & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} + \beta\, u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where λ^β is the first Robin eigenvalue of Ω.

We also call *Robin torsion function* of Ω the unique solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} + \beta\, u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We prove that the Robin ground state and the Robin torsion function of Ω are respectively log-concave and $\frac{1}{2}$-concave, provided the Robin parameter β exceeds a critical threshold. Such threshold depends on N, m, and on the geometry of Ω, precisely on the diameter and on the boundary curvatures up to order m.

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Nonsmooth nonoscillating generalized solution to forward and backward in time Cauchy problem for Kolmogorov–Fokker–Plank equation

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We consider the Cauchy problem for the Kolmogorov–Fokker–Plank equation

$$LU = -\varepsilon\frac{\partial U}{\partial t} + \varepsilon^2 A\frac{\partial^2 U}{\partial^2 x} - \varepsilon B\frac{\partial U}{\partial x} + \int (e^{-\varepsilon\nu\frac{\partial}{\partial x}} - 1)U\mu(d\nu) = 0. \tag{1}$$

with a small parameter $\varepsilon \to +0$.

We will consider a solution in the form

$$U = e^{-S(x,t)/\varepsilon}(\varphi(x,t) + O(\varepsilon)), \tag{2}$$

where $S \geqslant -C = const$. It is clear in this case that the pointwise limit

$$\lim_{\varepsilon\to+0}(-\varepsilon \ln U) = S(x,t)) \tag{3}$$

exists and, vice versa, the existence of limit (2) implies the representation similar to (1),

$$U = \exp\left(-\frac{S + o(1)}{\varepsilon}\right).$$

It is known that the function $S(x,t)$ is a solution to the Hamilton–Jacobi equation corresponding to (1) and, generally speaking, is not a smooth function. We introduce the notion of generalized exponential type solution $U(x,t)$ of (1):

1. The limit (3) exists and is a viscosity solution to the Hamilton–Jacobi equation corresponding to (1)
2. The weak (in a sense) limit is

$$w - \lim_{\varepsilon\to 0}(\varepsilon^{-1}Ue^{2S/\varepsilon}LU) = 0.$$

The last relation implies (see (2)) that $\varphi^2(x,t)$ is a delta-shock type solution to the continuity equation arising from the transport equation corresponding to (1) in the framework of the WKB–Maslov theory.

In the talk, I am going to discuss the property of generalized solutions introduced above and the possibility to solve the Cauchy problem backward in time for the solution of this type. For more details, see [1, 2].

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On a posteriory error estimates for a biharmonic obstacle problem

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We study a free boundary obstacle problem generated by the biharmonic operator. Mathematically this problem is formulated as follows: minimize the functional

$$J(v) = \int\limits_{\Omega} \left(\frac{1}{2}|\Delta v|^2 - fv \right) dx$$

over the closed convex set

$$\mathbb{K} = \left\{ v \in H^2(\Omega) : v\big|_{\partial\Omega} = 0,\ v \geqslant \varphi \text{ a.e. in } \Omega \right\}.$$

Here $\Omega \subset \mathbb{R}^n$ is an open, connected, and bounded domain with Lipschitz continuous boundary $\partial\Omega$, a given function $f \in L^2(\Omega)$, while φ is a given function (obstacle) such that $\varphi \in C^2(\overline{\Omega})$ and $\varphi \leqslant 0$ on $\partial\Omega$.

Such problem has an application in elasticity theory (frictionless equilibrium contact problems of free-supporting elastic plates or beams over a rigid obstacle).

Using the general theory developed for a wide class of convex variational problems, we deduce the error identity. One part of this identity characterizes the deviation of the function (approximation) from the exact solution of our variational problem, whereas the other is a fully computed value (it depends only on the data of the problem and known functions). In real life computations, this identity can be used to control the accuracy of approximate solutions.

Earlier the similar results were established in [1] for the case of rigidly fixed plates or beams.

The talk is based on results obtained in collaboration with D. E. Apushkinskaya.

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Optimal cyclic exploitation of distributed renewable resource with diffusion

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We consider a renewable resource distributed in a periodic environment, which we count as n-dimensional torus, where the dynamics of the resource is described by the Kolmogorov–Piskunov–Petrovsky–Fisher equation [1, 6, 7] in the divergent form

$$p_t = (\alpha(x)p_x)_x + a(x)p - b(x)p^2,$$

where $p = p(x, t)$ is the density of the resource at a point x of its distribution area at a time t, and the functions α, a, and b characterize the diffusion of the resource, the rates of its renewal, and the saturation of the environment with it, respectively. We assume that these functions continuously depend on a point of this area, but do not depend on time. In addition, it is assumed that the saturation rate b is positive and separated from zero by some constant $b_0 > 0$, the matrix α is positive definite, and its elements have derivatives satisfying the Holder condition with some positive exponent.

The resource is exploited by either the permanent harvesting or periodic impulse harvesting, or, in the case $n = 1$, by the harvesting control machine moving periodically along the circle and collecting at each moment a part of the resource density that depends on the current machine position and difficulties to search or extract the resource from this position.

Under reasonable assumptions we prove that for all harvesting modes there exists an admissible strategy which provide the maximum time averaged income in kind [2–5].

These studies were supported by Russian Science Foundation by project No. 19-11-00223.

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A characterization of the space of divergence-free vector fields from BMO based on the paradifferential operator calculus

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Let T be a singular integral operator of convolution type acting as a bounded operator in $L_2(\mathbb{R}^d)$. Under certain additional assumptions on T, its commutator

$$A_b = [b, T] \tag{1}$$

with a pointwise multiplier b is also bounded in $L_2(\mathbb{R}^d)$, provided that $b \in BMO(\mathbb{R}^d)$ (the fact is evident for $b \in L_\infty(\mathbb{R}^d)$), and the following estimate holds:

$$\|A_b\|_{L_2 \to L_2} \leqslant C_{d,T} \|b\|_{BMO}. \tag{2}$$

This well-known fact from harmonic analysis was generalized to a certain class of paradifferential operators A_b (paracommutators) depending on the coefficient b in a more general way than it is prescribed by (1). In particular, this applies to the operator of the form

$$A_b f = P(b \wedge Pf) \tag{3}$$

acting on vector fields $f \in L_2(\mathbb{R}^3; \mathbb{C}^3)$. Here b is a vector field in $\mathbb{R}^3$, $\wedge$ is the vector product in $\mathbb{C}^3$, P is the orthogonal projection on the space of vector fields in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ that are gradients of scalar functions.

The present talk concerns the converse of estimate (2), which allows one to majorize the BMO-norm of b in terms of A_b. Such estimates are known in the case where a linear mapping $b \mapsto A_b$ putting the paracommutator A_b in correspondence to a coefficient b, satisfies a certain nondegeneracy condition. However, to our knowledge, the existing results do not cover special case (3). Our goal is the estimate

$$\|b\|_{BMO} \leqslant C \|A_b\|_{L_2 \to L_2},$$

where A_b is given by (3). It can be shown that if b is a gradient of a scalar function, then $A_b = 0$. This implies that A_b depends only on the "divergence-free part" of b. We establish the last estimate in the case where $\operatorname{div} b = 0$.

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The Darboux theory of integrability for polynomial Liénard differential systems

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The Darboux theory of integrability is designed for finding and classifying integrable systems of polynomial ordinary differential equations. A collection of methods

available in the framework of this theory provides the necessary and sufficient conditions for a system to be integrable with a rational, a Darboux or a Liouvillian first integral [1]. These methods essentially use properties of invariants (algebraic and exponential) that the system has. The main difficulty in deriving algebraic invariants lies in the fact that the degrees of their generating polynomials are not known in advance. The aim of this talk is to present some novel results and generalizations in the Darboux theory of integrability. In particular, the problem of finding invariants for polynomial systems of ordinary differential equations in the plane will be discussed in details [2]. Along with this, we plan to consider the existence of non-autonomous Darboux first integrals of a special form.

Several applications of the Darboux theory will be presented. To be more precise, the following systems of first-order ordinary differential equations

$$x_t = y, \quad y_t = -f(x)y - g(x), \quad f(x), g(x) \in \mathbb{C}[x] \tag{1}$$

known as the Liénard differential systems will be considered. These systems describe oscillators with a polynomial damping $f(x)$ and a polynomial restoring force $g(x)$. Our main goal is to provide the solution of the Liouvillian integrability problem [3]. We shall prove that a generic non-linear Liénard differential system is not Liouvillian integrable. We shall demonstrate that for any numbers $\deg f$ and $\deg g$ such that $\deg g > \deg f$ there exist Liouvillian integrable sub-families. In addition, we plan to present some previously unknown Liouvillian integrable Liénard differential systems.

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On a class of transmission boundary value problems with integral boundary conditions

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In this work, we study the boundary value problem for linear second-order ordinary differential equations with discontinuous coefficients, in which we combine the weighted integral boundary conditions with the transmission conditions. We establish sufficient conditions guaranteeing that the resolvent has non maximal growth. Moreover, the operator studied generates an analytic semigroup with singularities in $L_p(0,1)$. The obtained results are then applied to the study of a nonlocal parabolic partial differential equation with regular boundary conditions.

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Effective Plancherel–Rotach type asymptotics of 2-D Hermitian orthogonal polynomials. The semiclassical approach

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The Hermitian-type orthogonal polynomials $H_{n_1,n_2}(z,a)$ are determined by the pair of recurrence relations for the polynomials $H_{n_1+1,n_2}(z,a)$, $H_{n_1,n_2+1}(z,a)$, $H_{n_1,n_2-1}(z,a)$, $H_{n_1,n_2-1}(z,a)$, $H_{n_1,n_2}(z,a)$. We obtain a uniform asymptotics of the diagonal polynomials $H_{n,n}(z,a)$ in the form of the Airy function for $n \gg 1$, which is a far-reaching generalization of the Plancherel–Rotach asymptotic formulas for Hermitian polynomials. In this talk, we discuss one of the possible approaches, which we call "real-valued semiclassics for asymptotics with complex-valued phases" (another approach based on the construction of decompositions of bases of homogeneous difference equations is discussed in the talk by A. I. Aptekarev and D. N. Tulyakov, both of these approaches can be found in [1]). Introducing an artificial small parameter $h = O(1/n)$ and a continuous function $\phi(x,z,a)$ such that $H(z,a)(z,a) = \phi(kh,z,a)$, we obtain a pseudodifferential equation for $\phi(x,z,a)$, where x is a variable and z,a are parameters. Seeking its solution in the WKB-form, one obtains the Hamilton–Jacobi equations with complex Hamiltonians connected with a third-order algebraic curve. This circumstance is the main difficulty of solving the problem and, as a rule, leads to the transition from the real variable x to the complex one. We propose a different approach based on a reduction of the original problem to three equations, two of which have asymptotics with a purely imaginary phase, and the symbol of the third one is pure real and has the form $\cos p + V_0(x) + hV_1(x) + O(h^2)$. This ultimately allows us to represent the desired asymptotics uniformly through the Airy function of the complex but real-valued argument. We also discuss how this approach is applied to obtain asymptotics for $H_{n_1,n_2}(z,a)$ based on the use of third-order ordinary differential equations.

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Homogenization of nonstationary periodic equations at the edge of a spectral gap

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In $L_2(\mathbb{R})$, we consider a second-order differential operator A_ε, $\varepsilon > 0$, given by the differential expression $A_\varepsilon = -\frac{d}{dx} g(x/\varepsilon) \frac{d}{dx} + \varepsilon^{-2} V(x/\varepsilon)$. Here g is a measurable function such that $0 < \alpha_0 \leqslant g(x) \leqslant \alpha_1 < \infty$, $g(x+1) = g(x)$, $x \in \mathbb{R}$, and $V \in L_1(0,1)$, $V(x+1) = V(x)$, $x \in \mathbb{R}$. We assume that $\inf \operatorname{spec} A = 0$, $A := A_1$.

It is well known that homogenization for the operator A_ε is a threshold effect near the edge of its spectrum. The spectrum of the operator A_ε has a band structure and may have gaps. Does it make sense to associate analogs of homogenization problems with the edges of internal gaps? We study this issue for a nonstationary Schrödinger equation and a hyperbolic equation involving the operator A_ε.

Let $\sigma > 0$ be a (non-degenerate) left edge of a band with odd number s in the spectrum of the operator A. Let $f, g \in L_2(\mathbb{R})$. Consider the Cauchy problems

$$\begin{cases} i\partial_t u_\varepsilon(x,t) = (A_\varepsilon u_\varepsilon)(x,t), \\ u_\varepsilon(x,0) = (\Upsilon_\varepsilon f)(x), \end{cases} \qquad \begin{cases} \partial_t^2 v_\varepsilon(x,t) = -(A_\varepsilon v_\varepsilon)(x,t) + \varepsilon^{-2}\sigma v_\varepsilon(x,t), \\ v_\varepsilon(x,0) = (\Upsilon_\varepsilon f)(x),\ (\partial_t v_\varepsilon)(x,0) = (\Upsilon_\varepsilon g)(x), \end{cases}$$

$$(\Upsilon_\varepsilon f)(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} (\Phi f)(k) \sum_{j=s}^{\infty} e^{ikx} \varphi_j(x/\varepsilon, \varepsilon k) \chi_{\widetilde{\Omega}_{j-s+1}}(\varepsilon k)\, dk.$$

Here $\{e^{ikx}\varphi_j(x,k)\}_{j=s}^{\infty}$ are the Bloch waves corresponding to the spectral bands of the operator A with numbers $j \geqslant s$, $\widetilde{\Omega}_j = (-j\pi, -(j-1)\pi] \cup ((j-1)\pi, j\pi]$, $j \in \mathbb{N}$, are the Brillouin zones, and $(\Phi f)(k)$ is the Fourier image of a function $f(x)$. We prove the following estimates:

$$\|u_\varepsilon(\cdot,t) - e^{-it\varepsilon^{-2}\sigma}\varphi_\sigma(\cdot/\varepsilon)u_0(\cdot,t)\|_{L_2(\mathbb{R})} \leqslant C(1+|t|^{1/2})\varepsilon\|f\|_{H^2(\mathbb{R})}, \qquad f \in H^2(\mathbb{R}),$$

$$\|v_\varepsilon(\cdot,t) - \varphi_\sigma(\cdot/\varepsilon)v_0(\cdot,t)\|_{L_2(\mathbb{R})} \leqslant C(1+|t|^{1/2})\varepsilon(\|f\|_{H^{3/2}(\mathbb{R})} + \|g\|_{H^{1/2}(\mathbb{R})}),$$
$$f \in H^{3/2}(\mathbb{R}),\ g \in H^{1/2}(\mathbb{R}),$$

where u_0 and v_0 are the solutions of the effective problems

$$\begin{cases} i\partial_t u_0(x,t) = (A_\sigma^{\mathrm{hom}} u_0)(x,t), \\ u_0(x,0) = f(x), \end{cases} \qquad \begin{cases} \partial_t^2 v_0(x,t) = -(A_\sigma^{\mathrm{hom}} v_0)(x,t), \\ v_0(x,0) = f(x),\ (\partial_t v_0)(x,0) = g(x), \end{cases}$$

$A_\sigma^{\mathrm{hom}} = -b_\sigma \frac{d^2}{dx^2}$, $b_\sigma > 0$ is the coefficient in the asymptotics of the band function $E(k) = E_s(k)$: $E(k) \sim \sigma + b_\sigma k^2$, $k \sim 0$; and $\varphi_\sigma(x) = \varphi_s(x,0)$ is the periodic solution of the equation $A\varphi_\sigma = \sigma\varphi_\sigma$, normalized in $L_2(0,1)$.

These results are sharp with respect to the norm type as well as the dependence on t. The other edges of the spectral gaps have also been studied. The results are published in [1].

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Asymptotics for Volterra equations with advanced variable

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Working over parking problem [1], we arrived at the integral equation

$$f(x+1) = \frac{n+1}{x} \int_0^x f(t)\,dt + p(x+1),\ x > 0,\ n = 0, 1, 2, \ldots, \tag{1}$$

where $p(t)$ is a continuous function satisfying a certain convergence property. Dvoretzky and Robbins [2] showed the existence of the limit $\lim\limits_{x\to\infty} f(x)$ for $n = 1$ and estimated it.

1. We show that for the homogeneous equation with $p \equiv 0$ and integer $n \geqslant 0$, there exists a unique n-degree monic polynomial solution $q_n(x)$ with positive coefficients. Assuming that $p_i = \sup\limits_{i \leqslant t \leqslant i+1} |p(t)|$ and $\sum\limits_{i=2}^{\infty} i^{-n} p_i < \infty$, we prove that there exists a constant λ such that $\sup\limits_{m+1 \leqslant x \leqslant m+2} |f(x) - \lambda q_n(x)| \to 0$ as $m \to \infty$.

2. Under similar entry conditions for the more general equation

$$f(x+1) = \frac{\int_0^x (K(t) + K(x-t)) f(t)\,dt}{\int_0^x K(t)\,dt} + p(x+1) \tag{2}$$

with $K(t) > 0$, we prove $f(x) = O(x)$. If the integral kernel $K(t)$ is polynomial, then, in addition, the limit $\lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ exists.

3. Also, we consider the equation

$$f(x+1) = \frac{\int_0^x K(t)f(t)\,dt}{\int_0^x K(t)\,dt} + p(x+1) \tag{3}$$

with $\sum\limits_{i=2}^{\infty} p_i < \infty$. We prove that the solution $f(x)$ is bounded and, under extra conditions on the kernel $K(t)$, has a limit $\lambda = \lim\limits_{x\to\infty} f(x)$. In terms of the estimate for $T_n = \sup\limits_{n\leqslant t\leqslant n+1} |f(t) - \lambda| \to 0$ as $n \to \infty$, we obtain the rate of convergence.

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Korteweg–de Vries equation on the Uhlenbeck manifold

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We consider the family

$$-y" + p(x)y = \lambda y; \quad y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0 \tag{1}$$

of periodic eigenvalue problems with 2π-periodic real potential

$$p \in P := \{C^0(2\pi) \ \wedge \int_0^{2\pi} p(x)dx = 0\}$$

as a functional parameter. We consider the Uhlenbeck manifold of all eigenfunctions [1]

$$Y := \{y \in C^2(2\pi) : \int_0^{2\pi} y^2 dx = 1 \ \wedge \ \exists (p,\lambda) \in P \times R : (1) \ is \ true\}.$$

An analytic and topological description of its foliation by hypersurfaces defined by the condition that the n-th spectral lacuna has fixed oriented length is given (see [2, 3]). Then we uplift the KdV equation from the space of potentials P to the Uhlenbeck manifold of eigenfunctions Y (see [4]).

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On some skew products on simplest multidimensional manifolds

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We study here the returnability properties of trajectories of skew products with zero-topological entropy defined on n-dimensional ($n \geqslant 2$) cells, cylinders, and tori [1].

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Nonlocal abstract Cauchy problem for nonlinear fractional integral equation

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This note is devoted to the study of the following nonlocal abstract Cauchy problem:

$$u(t) = u_0 + \sum_{i=1}^{r} c_i u(t_i) + \frac{1}{\Gamma(\alpha)} \int_0^t [\psi(t) - \psi(s)]^{\alpha-1} [Au(s) + f(s, B(s)u(s))] ds,$$

where u_0 is a given element of a Banach space E, $c_1, \ldots, c_r$ are real numbers, $t_1, \ldots, t_r \in J = [0, T]$, $T > 0$, A is a closed linear operator defined on a dense set S_1 in E, $\{B(t) : t \in J\}$ is a family of closed linear operators defined on a set $S_2 \supset S_1$, f is a nonlinear function defined on $J \times E$ and with values in E, $0 < \alpha \leqslant 1$, ψ is a real function defined on J, and Γ is the gamma function. It is supposed that A generates an analytic semigroup. The solution of the considered equation is given under suitable conditions. Some properties are studied and application on the nonlocal Cauchy problem is given for fractional integro-partial differential equations.

A hybrid stochastic fractional order coronavirus mathematical model via the reservoir–people transmission network

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Mathematical models of epidemiological systems enable studying and predicting about potential spread of disease. We develop and analyze a mathematical model to simulate the coronavirus transmission dynamics based on reservoir–people transmission network. Assuming that $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is a complete probability space and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0,\ 1 \leqslant i \leqslant d\}$, consider the following d-dimensional stochastic differential equation with the conditions expressed in [1, 3]:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \quad X(t_0) = X_0, \quad t \geqslant t_0.$$

We consider an epidemic model based on a 6-dimensional stochastic fractional differential equation. This model is the simplified and normalized form of the model presented in [2]. Let S_p and E_p refer to the number of susceptible and exposed people, I_p denote the number of symptomatic infected people, A_p denote the number of asymptomatic infected people, R_p denote the number of removed people including recovered and dead people, and W denote the COVID in reservoir. Therefore, the perturbed fractional order system can be described by using the Caputo–Fabrizio fractional derivative as follows:

$$\begin{aligned}
{}^{CFC}D^n_t s_p &= (n_p - m_p s_p - s_p b_p(i_p + k a_p) - s_p w b_w)dt + \sigma_1 dB_1 - s_p \sigma_2 dB_2 \\
&\quad - s_p(i_p + k a_p)\sigma_3 dB_3 - s_p w \sigma_4 dB_4, \\
{}^{CFC}D^n_t e_p &= (s_p b_p(i_p + k a_p) + b_w s_p w - w_p e_p + \delta_p w_p e_p - \delta_p w'_p e_p - m_p e_p)dt \\
&\quad + s_p \sigma_3 (i_p + k a_p) dB_3 + s_p w \sigma_4 dB_4 + w_p e_p \sigma_5 dB_5 - w'_p e_p \sigma_5 dB_5 \\
&\quad - e_p \sigma_2 dB_2, \\
{}^{CFC}D^n_t i_p &= (w_p e_p - \delta_p w_p e_p - \gamma_p i_p - i_p m_p)dt - i_p \sigma_2 dB_2 - \sigma_5 w_p e_p dB_5, \\
{}^{CFC}D^n_t a_p &= (\delta_p w'_p e_p - \gamma'_p a_p - m_p a_p)dt - a_p \sigma_2 dB_2, \\
{}^{CFC}D^n_t r_p &= (\gamma_p i_p + \gamma'_p a_p - r_p m_p)dt - r_p \sigma_2 dB_2, \\
{}^{CFC}D^n_t w &= \varepsilon(i_p + c a_p - w)dt + \sigma_6(i_p + c a_p - w)dB_6.
\end{aligned} \tag{1}$$

Theorem 1. *For any given initial value* $(s_p(0), e_p(0), i_p(0), a_p(0), r_p(0), w(0)) \in \mathbb{R}^6_+$, *there exists the solution* $(s_p(t), e_p(t), i_p(t), a_p(t), r_p(t), w(t))$ *for model* (1) *and the solution will remain in* $\mathbb{R}^6_+$ *with probability one.*

Theorem 2. *The coefficients of differential equation* (1) *are locally Lipschitz.*

Also, the simulation is carried out to demonstrate efficiency of our model, and the possibility of comparing the stochastic fractional model with the deterministic model, using data extracted from WHO reports.

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On generation of resolving family of operators for distributed order differential equations

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Let $\mathcal{Z}$ be a Banach space, D_t^β is the Riemann–Liouville derivative for $\beta > 0$ and the Riemann–Liouville integral for $\beta < 0$, $A \in \mathcal{C}l(\mathcal{Z})$, i. e. it is a closed linear operator with domain D_A dense in $\mathcal{Z}$, $-\infty < b \leqslant 0 \leqslant m-1 < c \leqslant m \in \mathbb{N}$, $\omega \in L_1(b,c)$, $z_k \in \mathcal{Z}$, $k = 0, 1, \ldots, m-1$. Consider the initial value problem

$$\int\limits_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \ldots, m-1, \tag{1}$$

$$\int\limits_{b}^{c} \omega(\alpha) D_t^{\alpha} z(t) d\alpha = Az(t), \ \ t > 0. \tag{2}$$

Let $S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta\}$, and $A \in \mathcal{C}l(\mathcal{Z})$ be such that

$$\exists \theta_0 \in (\pi/2, \pi], a_0 \geqslant 0 \ \forall \lambda \in S_{\theta_0,a_0} \ W(\lambda) \in \rho(A);$$

$$\exists \varepsilon_0 > 0 \ \forall \varepsilon \in (0, \varepsilon_0), \theta \in (\pi/2, \theta_0), \ a > a_0 \ \exists K(\theta, a, \varepsilon) > 0 \ \forall \lambda \in S_{\theta,a}$$
$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant K(\theta, a, \varepsilon)|\lambda|^{1-m}|\lambda - a|^{m-c-1+\varepsilon}.$$

We write $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$ in this case, see [1].

A family of operators $\{S_l(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called *l-resolving*, $l \in \{0, \ldots, m-1\}$, for equation (2) if the following conditions are satisfied:

(i) $S_l(t)$ is strongly continuous at $t > 0$;

(ii) $S_l(t)[D_A] \subset D_A$, $S_l(t)Ax = AS_l(t)x$ for all $x \in D_A$, $t > 0$;

(iii) for every $z_l \in D_A$, $S_l(t)z_l$ is a solution of problem (1), (2) with $z_k = 0$, $k \in \{0, \ldots, m-1\} \setminus \{l\}$.

Theorem 1. *Let $-\infty < b \leqslant 0 \leqslant m-1 < c \leqslant m \in \mathbb{N}$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geqslant 0$, $\omega \in L_1(b,c)$. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists an analytic 0-resolving family of operators of type $(\theta_0 - \pi/2, a_0, m+\varepsilon-c)$ for equation (2) if and only if $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$. In this case, there exist analytic k-resolving families of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, $k = 1, 2, \ldots, m-1$, for equation (2). For every $k = 0, 1, \ldots, m-1$, a k-resolving family of operators is unique, $S_k(t) \equiv J_t^k S_0(t)$, $t > 0$.*

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Application of the cascade search principle for zeros of functionals to the solution of one-parametric families of equations with multivalued operators

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The talk is based on the joint work with Yu. N. Zakharyan.

Let us consider a family of non-negative multivalued functionals $\{\Phi_t\}_{t\in[0;1]}$, where $\Phi_t : \overline{U} \rightrightarrows \mathbb{R}_+$, $t \in [0;1]$, $U \subset X$ is an open subset in a metric space (X, d). Suppose that Φ_0 has zero in U, i.e., there is a point $x_0 \in U$ such that $0 \in \Phi_0(x_0)$. What are sufficient conditions to provide the zero existence preservation for the functionals Φ_t in the subset U, at every $t \in [0;1]$?

To answer this question, we use the following minor modification of the concept of (α, β)-search multivalued functional $(0 \leqslant \beta < \alpha)$ suggested by the author [1].

Definition 1. A multivalued functional $\Phi_t : \overline{U} \rightrightarrows \mathbb{R}_+$ is said to be (α, β)-*search on* $\overline{U}$ iff for any $x \in \overline{U}$, $r > 0$, and $c \in \Phi_t(x)$, such that $B(x; r) \subseteq \overline{U}$ and $c \leqslant (\alpha - \beta)r$, there exists a point $x' \in \overline{U}$ and $c' \in \Phi(x')$ such that $d(x, x') \leqslant \frac{1}{\alpha}c$ and $c' \leqslant \frac{\beta}{\alpha}c$.

Definition 2. The graph $Graph(\Phi_t) := \{(x, c) | x \in \overline{U}, c \in \Phi_t(x)\}$ of the functional Φ_t is called $\{0\}$-*complete* iff any fundamental sequence $\{(x_n, c_n)\} \subseteq Graph(\Phi_t)$ with $c_n \to 0$ converges (in the sense of component-wise convergence) to some element $(\xi, 0) \in Graph(\Phi_t)$.

In 2009–2013, the author suggested several versions of the cascade search principle for zeros of functionals. In particular, its generalized local version is given in [1].

On the base of these concepts and statements the following result is proved.

Theorem 1. *Let (X, d) be a metric space, $U \subset X$ be an open subset, and $\theta : [0;1] \to \mathbb{R}$ be a continuous increasing function. Let $\Phi = \{\Phi_t : \overline{U} \rightrightarrows \mathbb{R}_+\}_{t\in[0;1]}$ be a parametrized family of multivalued functionals which are (α, β)-search on $\overline{U}$, with $\{0\}$-complete graphs. In addition, let the family Φ be θ-continuous, i.e., for any $t', t'' \in [0;1]$, $x \in \overline{U}$, and for every $c' \in \Phi_{t'}(x)$ there exists $c'' \in \Phi_{t''}(x)$ such that $|c' - c''| \leqslant |\theta(t') - \theta(t'')|$. Suppose that $0 \notin \Phi_t(x)$ for any $x \in \partial U$ and $t \in [0;1]$. Then, if there is $x_0 \in U$ such that $0 \in \Phi_0(x_0)$, then there is $x_1 \in U$ such that $0 \in \Phi_1(x_1)$.*

This result implies several corollaries on the solvability of one-parametric families of inclusions and equations such as $x \in F_{1t}(x) \cap \cdots \cap F_{mt}(x)$, $\bigcap\limits_{k=1}^{m} F_{kt}(x) \neq \varnothing$, where

F_{kt} is a multivalued operator defined on an open subset of a metric space, $1 \leqslant k \leqslant m, t \in [0;1]$. In particular, the known theorem by M. Frigon and A. Granas [2, 3] on the fixed point preservation for a contracting family of multivalued self-mappings is one of the corollaries.

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Global bifurcation analysis and applications of multi-parameter dynamical systems

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We carry a global bifurcation analysis of multi-parameter polynomial dynamical systems. To control all their limit cycle bifurcations, especially, bifurcations of multiple limit cycles, it is necessary to know the properties and combine the effects of all their rotation parameters. This can be done by developing the new bifurcation geometric methods based on Perko's planar termination principle [1]. This principle is a consequence of the principle of natural termination stated for higher-dimensional dynamical systems by A. Wintner who studied one-parameter families of periodic orbits of the restricted three-body problem and used Puiseux series to show that in the analytic case any one-parameter family of periodic orbits can be uniquely continued through any bifurcation except a period-doubling bifurcation. Such a bifurcation can happen, e. g., in the three-dimensional Lorenz system. But this cannot happen for planar systems. That is why the Wintner–Perko termination principle is applied for studying multiple limit cycle bifurcations of planar polynomial dynamical systems [1]. If we do not know the cyclicity of the termination points, then, applying canonical systems with field rotation parameters, we use geometric properties of the spirals filling the interior and exterior domains of limit cycles. Applying this approach, we have solved, e. g., *Hilbert's Sixteenth Problem* on the maximum number and distribution of limit cycles for the general Liénard polynomial system with an arbitrary number of singular points [2], the Euler–Lagrange–Liénard polynomial mechanical system [3], Leslie–Gower systems modeling the population dynamics in real ecological or biomedical patterns [4], and the reduced planar quartic Topp system modeling the dynamics of diabetes [5]. Finally, applying a similar approach, we have considered various applications of three-dimensional polynomial dynamical systems and, in particular, completed the strange attractor bifurcation scenario in Lorenz type systems globally connecting the homoclinic, period-doubling, Andronov–Shilnikov, and period-halving bifurcations of its limit cycles [6].

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Nonexistence of solutions for some nonlinear inequalities with transformed arguments in a half-space

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We establish nonexistence of different types of solutions for some semilinear elliptic inequalities with a transformed argument in a half-space. The proofs are based on the test function method.

On the Rayleigh–Taylor instability

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We consider the inhomgeneous incompressible Euler equations describing two fluids with different constant densities under the influence of gravity. Initially the fluids are supposed to be at rest and separated by a flat horizontal interface with the heavier fluid being on top of the lighter one. Due to gravity this configuration is unstable, the two fluids begin to mix in a more and more turbulent way. This process is called the Rayleigh–Taylor instability. In the talk we will discuss the existence of solutions to the Euler equations which reflect a turbulent mixing of the two fluids in a quadratically growing zone.

This is based on joint works with József Kolumbán and László Székelyhidi.

Global existence of solutions of semilinear parabolic equation with nonlinear memory condition

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We investigate the global solvability and blow-up in finite time for the semilinear heat equation with the nonlinear memory boundary condition

$$u_t = \Delta u + c(t)u^p \ \text{ for } \ x \in \Omega, \ \ t > 0, \tag{1}$$

$$\frac{\partial u(x,t)}{\partial \nu} = k(t) \int_0^t u^q(x,\tau)\, d\tau \ \text{ for } \ x \in \partial\Omega, \ \ t > 0, \tag{2}$$

$$u(x,0) = u_0(x) \ \text{ for } \ x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geqslant 1$ with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, $p > 0$ and $q > 0$. Here $c(t)$ and $k(t)$ are nonnegative continuous functions for $t \geqslant 0$. The initial datum $u_0(x)$ is a nonnegative $C^1(\overline{\Omega})$ function satisfying the boundary condition at $t = 0$.

We prove global existence of solutions of (1)–(3).

Theorem 1. *There are no nontrivial global solutions of* (1)–(3) *if*

$$p > 1 \ \text{ and } \ \int_0^\infty c(t)\, dt = \infty$$

or

$$q > 1, \ \ k(t) \geqslant \underline{k}(t) \geqslant 0 \ \text{ and } \ \int_0^\infty t\underline{k}(t)\, dt = \infty$$

and at least one of the following conditions is fulfilled:

$$\underline{k}(t) \leqslant \frac{c}{t^2} \ \text{ for large values of } t \ (c > 0)$$

or

$$t^{1-q}\underline{k}(t) \ \text{ is nonincreasing for large values of } t.$$

To formulate the global existence result for problem (1)–(3), we suppose that

$$\int_0^\infty (c(t) + tk(t))\ dt < \infty \tag{4}$$

and there exist positive constants α, t_0, and K such that $\alpha > t_0$ and

$$\int_{t-t_0}^t \frac{\tau k(\tau)}{\sqrt{t-\tau}}\, d\tau \leqslant K \ \text{ for } \ t \geqslant \alpha. \tag{5}$$

Theorem 2. *Let* $\min(p,q) > 1$ *and* (4), (5) *hold. Then problem* (1)–(3) *has bounded global solutions for small initial data.*

Similar results we obtain for the case $p = 1, q > 1$.

The results of the talk have been published in [1].

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On the relationship of solutions of singular equations with fractional powers of the operator coefficient of the equation

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For $k > 0$, the Cauchy problem for the Euler–Poisson–Darboux (EPD) equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \tag{1}$$

$$u(0) = u_0, \quad u'(0) = 0 \tag{2}$$

in a Banach space E is well-posed only if the operator $A \in G_k$ is a generator of the Bessel operator function (OFB) $Y_k(t; A)$ or, which is the same, $k/2$ times integrated cosine operator function. Moreover, if $k \geqslant 1$, then the initial condition of the form $u'(0) = 0$ holds automatically, but it is possible to set a non-zero second condition with a weight. The question arises: is it possible to weaken the conditions on the class of operators A so that some other initial value problems would also be solvable? While B is the generator of the semigroup $T(t; B),\ t \geqslant 0$, the operator $A = B^2$ is not necessarily an OFB generator, i.e., problem (1), (2) may not be well-posed. Is it possible to consider another solvable problem for such an operator? Let's try to find a solution to equation (1) going to zero as $t \to \infty$ and consider the case of the so-called incomplete initial problem where the second initial condition for $t = 0$ is not specified precisely because the solution tends to zero as $t \to \infty$. Wherein the form of the first initial condition will depend on the parameter $k > 0$ from equation (1). The indicated tendency of the solution to zero is provided by the form of the operator A. We will assume that $A = B^2$, where B is the generator of the C_0-semigroup $T(t; B)$ admitting the estimate $\|T(t; B)\| \leqslant \Upsilon e^{-\omega t},\ \Upsilon \geqslant 1,\ \omega > 0$. Finding a solution to this problem, one uses fractional powers of the operator B, which is a distinctive feature of the considered incomplete initial problems, and also the operator function

$$M_k(t; B)v_0 = \frac{\sqrt{\pi}}{\Gamma(k/2)} \int_1^\infty (s^2 - 1)^{k/2-1} T(ts; B)v_0 \, ds, \quad v_0 \in E$$

called the McDonald operator function by the author.

When finding a solution to the incomplete initial problem for the Legendre equation

$$u''(t)(t) + k \operatorname{cth} t \; u'(t) + (k/2)^2 u(t) = Au(t), \quad t > 0,$$

fractional powers of the operator B and the associated Legendre operator function

$$Q_k(t; B)v_0 = \frac{\sqrt{\pi}}{\Gamma(k/2)} \operatorname{sh}^{1-k} t \int_t^\infty (\operatorname{ch} s - \operatorname{ch} t)^{k/2-1} T(s; B)v_0 \, ds, \quad \omega > |k/2 - 1|, \quad v_0 \in E$$

are also used.

Note that the fractional powers of the operator coefficient included in the equation are used not only for constructing solutions to incomplete initial problems, they also connect the Dirichlet and the weighted Neumann conditions imposed on solutions of the boundary value problems for the EPD equation in the elliptic case.

Partial regularity in time for the Landau equation

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The Landau equation is a kinetic equation proposed by L. D. Landau in 1936 to describe collisions between charged particles interacting via the Coulomb potential. It takes the form of a nonlinear parabolic equation with a nonlocal dependence of the coefficients in the solution. Whether classical solutions of the Landau equation exist for all times or blow up in finite time remains a major open question in the mathematical theory of kinetic models at the time of this writing. The purpose of this talk is to present a recent result showing that the Hausdorff dimension of singular times of weak solutions to the Landau equation satisfying the physically natural a priori estimates cannot exceed 1/2. (Work in collaboration with M. P. Gualdani, C. Imbert, and A. Vasseur.)

On the coexistence of hyperbolic basic sets of dynamical systems

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According to S. Smale, the hyperbolic non-wandering set $NW(f)$ of a diffeomorphism $f : M^n \to M^n$ with dense set of periodic points (M^n is a closed smooth manifold) is represented as a finite union of invariant closed sets, each of which contains a transitive orbit. Such diffeomorphisms are called A-diffeomorphisms and the invariant sets are called basic.

If the dimension of some basic set Λ of an-A-diffeomorphism f is greater than one and coincides with the dimension of the supporting manifold M^n ($n > 1$), then Λ is unique and coincides with the whole manifold M^n. In this case, the diffeomorphism f is an Anosov diffeomorphism. It follows from works of J. Franks and S. Newhouse, that if the dimensions of unstable or stable manifolds of points from Λ are equal to one, then the ambient manifold M^n is n-torus.

If the dimension of a basic set Λ of a diffeomorphism f is $n-1$, then, according to R. Plykin, Λ is either an attractor or a repeller. The author of the report and E. Zhuzhoma have proved that if the non-wandering set of a structurally stable diffeomorphism f contains an orientable attractor (repeller) whose dimension is $n-1$ and coincides with the dimension of the unstable (stable) manifolds of its points (such attractor (repeller) is called expanding (contracting)), then the ambient manifold M^n is n-torus [1]. Recently, the topological structure of manifolds M^n ($n \geqslant 3$) admitting

diffeomorphisms whose non-wandering set consists of orientable expanding attractor and contracting repellers of dimension $n-1$, was completely described by the author in collaboration with E. Zhuzhoma and V. Medvedev.

If $n=2$, then there is a diffeomorphism f on a surface M^2 of any genus whose non-wandering set contains one-dimensional basic sets. In recent papers by the author, E. Kurenkov and D. Mints [2–4], the topological structure of their embedding in ambient surface was investigated and dynamics of f in connection with the properties of basic sets and genus of ambient surface was studied.

For $n=3$, we consider a class of diffeomorphisms assuming that all basic sets of a diffeomorphism are two-dimensional. As it was mentioned, such basic sets are either attractors or repellers and there are only two types of such basic sets: a basic set of the first type is homeomorphic to two-dimensional torus which is tamely embedded in the ambient manifold and such basic set is called a surface; a basic set of the second type is locally arranged as direct product of Cantor set and open two-dimensional disk (it is an expanding attractor or a contracting repeller in this case). It is shown by the author, M. Barinova and O. Pochinka in [5], that the non-wandering set of any diffeomorphism from the considered class consists of either basic sets of the first type or basic sets of the second type.

In the report, we will discuss the above results from the point of view of their application to the topological classification of cascades with the hyperbolic non-wandering set on manifolds. For an overview of the topic, see books and survey [6–8].

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Existence and multiplicity of solutions for a nonlocal problem with critical Sobolev–Hardy nonlinearities

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We investigate the following nonlocal equation involving critical Sobolev–Hardy exponents:

$$(\mathrm{P})\begin{cases}(-\Delta)^s u-\mu\frac{u}{|x|^{2s}}=\lambda|u|^{q-2}u+\frac{|u|^{2^*_\alpha-2}u}{|x|^\alpha} & \text{in } \Omega,\\ \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0<s<1$, $\lambda>0$ is a parameter, $0\leqslant\mu<\mu_0$ with $\mu_0=4^s\dfrac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}$ being the sharp constant of the fractional Sobolev–Hardy in $\mathbb{R}^N$, $0<\alpha<2s<N$, $1<q<2^*_s$, where $2^*_s=\frac{2N}{N-2s}$ and $2^*_\alpha=\frac{2(N-\alpha)}{N-2s}$ are the fractional critical Sobolev and Sobolev–Hardy exponents respectively. The fractional Laplacian $(-\Delta)^s$ with $s\in(0,1)$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)^s u(x)=-\frac{1}{2}\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}dy\quad(x\in\mathbb{R}^N).$$

We combine several variational methods in order to show the existence of multiple positive solutions to our problem. Precisely, we prove the following result:

Theorem 1. *For every $\mu\in[0,\mu_0)$, there exists $\Lambda\in(0,\infty)$ such that*

(*i*) $\forall\ \lambda\in(0,\Lambda)$, *problem* (P) *has two distinct solutions.*

(*ii*) *For* $\lambda=\Lambda$, *problem* (P) *has at least one solution.*

(*iii*) $\forall\ \lambda\in(\Lambda,\infty)$, *problem* (P) *has no solution.*

Among the tools we used we cite mountain pass theorems, sub and super solutions, min-max methods and other basic notions of the critical point theory.

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Smoothness of generalized solutions of the second boundary-value problem for differential-difference equations on an interval of non-integer length

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We consider the second boundary-value problem for the second order differential–difference equation with variable coefficients on the finite interval $(0, d)$, $d = N+\theta$, $0 < \theta < 1$. We assume that the Hermitian part of the difference operator is a positive definite operator. We have proved that the corresponding differential–difference operator is a Fredholm operator. It is shown that a generalized solution is smooth within subintervals obtained by deleting the orbits for the ends of the interval $(0, d)$, generated by the group of integer shifts. However, smoothness of generalized solutions can be violated at the points of the mentioned orbits. We prove that, if the right-hand side of the equation is orthogonal in $L_2(0, d)$ to a finite number of linearly independent functions, then a generalized solution from the Sobolev space $W_2^1(0, d)$ belongs to the space $W_2^2(0, d)$.

Similar results were obtained for the first boundary-value problem in [1].

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Elliptic differential-difference equations with finite and infinite boundaries traces

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We consider the boundary value problem

$$A_R u = -\sum_{i,j=1}^{n} (R_{ij} u_{x_j})_{x_i} = f(x) \quad (x \in Q), \tag{1}$$

$$u(x) = 0 \quad (x \notin Q). \tag{2}$$

Here Q is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂Q, $f \in L_2(Q)$, the difference operators $R_{ij} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ are given by

$$R_{ij} u(x) = \sum_{h \in M_{ij}} a_{ijh} (u(x+h) + u(x-h)) \quad (\ a_{ijh} \in \mathbb{R}),$$

where $M_{ij} \subseteq M$ is a finite set of vectors with incommensurable coordinates. The solution u of boundary value problem (1), (2) belongs to the Sobolev space $\overset{\circ}{H}{}^1(Q)$ (by a solution, we mean the generalized solution understood in the standard way).

For elliptic differential-difference equations with integer or commensurable shifts of the independent variables, the theory of boundary-value problems is created in the works of Skubachevskii (see [1]).

Unlike the problems studied in [1], equation (1) contains incommensurable shifts of the arguments, which greatly complicates the study. However, in the case where the orbit of the boundary ∂Q under the shifts presented the difference operator, is finite, the methods developed for problems with integer shifts are applicable. In particular, problem (1)-(2) can be reduced to the boundary problem for a differential equation with nonlocal boundary conditions.

For the case where the orbit of the boundary under the shifts presented in the difference operator, is infinite, the nature of the problem changes fundamentally. In particular, its solutions can have an almost everywhere dense set of derivative discontinuites. A method for obtaining the conditions of strong ellipticity (the fulfillment of the Gårding-type inequality) based on the construction of a system of interrelated matrix polynomials is proposed [1]. These conditions are stable relative to small perturbations of the shifts of the difference operator and allow us to approximate the problems under consideration by problems with rational (commensurable) shifts.

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On elliptic complexes in relative elliptic theory

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We study the theory of complexes in relative elliptic theory. This theory was introduced by Sternin [1, 2] as a theory of boundary-value problems with conditions on submanifolds of arbitrary dimension.

Here we consider pairs (M, X), where M is a closed smooth manifold with its submanifold X of codimension ν, $i : X \hookrightarrow M$ denotes the corresponding embedding.

We deal with complexes of bounded operators acting in Sobolev spaces

$$0 \to \begin{matrix} H^{s_0}(M, E_0) \\ \oplus \\ H^{t_0}(X, F_0) \end{matrix} \xrightarrow{d_0} \begin{matrix} H^{s_1}(M, E_1) \\ \oplus \\ H^{t_1}(X, F_1) \end{matrix} \xrightarrow{d_1} \begin{matrix} H^{s_2}(M, E_2) \\ \oplus \\ H^{t_2}(X, F_2) \end{matrix} \xrightarrow{d_2} \dots \xrightarrow{d_{m-1}} \begin{matrix} H^{s_m}(M, E_m) \\ \oplus \\ H^{t_m}(X, F_m) \end{matrix} \to 0, \tag{1}$$

where

- E_j, F_j are complex vector bundles on M and X respectively, H^s are Sobolev spaces of vector bundle sections;
- the operators d_j are morphisms in the sense of [3],

$$d_j = \begin{pmatrix} A_j & C_j \\ B_j & D_j \end{pmatrix},$$

where A_j and D_j are pseudodifferential operators (ψDOs below) on M and X respectively. Boundary and coboundary operators B_j and C_j are equal to $B_j = D''_{X,j} i^* D''_{M,j}, C_j = D'_{M,j} i_* D'_{X,j}$ for some ψDOs $D'_{M,j}, D''_{M,j}$ and $D'_{X,j}, D''_{X,j}$ on M and X respectively. Boundary operator i^* is defined as

$$i^* : H^s(M,E) \longrightarrow H^{s-\nu/2}(X, E|_X), \quad i^* : u \longmapsto u|_X, \quad s > \nu/2.$$

A dual coboundary operator i_* is defined as

$$i_* : H^{-s+\nu/2}(X, E|_X) \longrightarrow H^{-s}(M,E), s > \nu/2.$$

For such complexes we introduce the notion of ellipticity. The following result is obtained.

Theorem 1. *If the complex* (1) *is elliptic, then it has Fredholm property.*

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Fixed point results of Reich contraction in fuzzy metric spaces endowed with graph

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In this paper, we define a new class of Reich type contractions in the framework of complete fuzzy metric spaces satisfying the graph preserving conditions. A large number of different types of contractive mappings formulated using directed graphs in literature satisfy the presented contractive condition. Our main result is a natural generalization from fuzzy metric spaces to fuzzy metric spaces with a graph and enriches our knowledge of fixed points in such spaces. The results are further validated with the examples and application.

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Nonlinear elliptic equations of nonstrictly divergent form and subcoercive operators

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We consider nonlinear elliptic equations and systems of the form

$$\operatorname{div}^t A(x, D^s u) = f(x)$$

in $\mathbb{R}^n$, $s+t$ is even. We assume that A satisfies the structure condition

$$|B(x,\xi) - B(x,\eta)| \leqslant |\xi - \eta|,$$

B is defined by $\Delta^{(s+t)/2}u + \operatorname{div}^t B(x, D^s u) = \varkappa \operatorname{div}^t A(x, D^s u)$, $\varkappa > 0$ is a normalizing factor (say, 'a perturbation of the poly-Laplacian should be Lipschitz-continuous with constant 1'), and $A(x,0) = 0$. Under the stronger condition $|B(x,\xi) - B(x,\eta)| \leqslant K|\xi - \eta|$, $K < 1$, the operator is coercive in pair with $\Delta^{(s-t)/2}u$ in H^s. In the case $s = t$, the condition with $K < 1$ coincides with the standard structure condition for divergent equations and systems, and in the case of a single nondivergent ($t = 0$) equation with Cordes condition. Whereas the condition with $K = 1$ allows degeneration of ellipticity, e.g., the operator $\Delta^m u - |D^{2m}u|$ as well as the linear operator $\Delta^{(s+t)/2}u - \partial_{x_1}^{s+t}u$ satisfies it. And for $K = 1$ our operator is still monotone in H^s in pair with $\Delta^{(s-t)/2}u$.

The nonstrictly divergent case $s \neq t$ (in contrast to the strictly divergent case $s = t$) allows one to establish some estimates of solutions even under degenerate structure condition, namely,

$$\|D^{s-1}u\|_{a-2} \leqslant c_a \, \|I_{t-1} f\|_{a+2}$$

for some range of a depending on s, t, n. Here $\|\cdot\|_a$ is the norm in $L_2(\mathbb{R}^n)$ with the power weight $(1+|x|)^a$, and I_t is a Riesz potential of order t.

We will discuss existence and uniqueness results in various settings under more or less strong restrictions: for solutions in the sense of integral identity, in the sense of maximal monotone extension, in the sense of generalized pseudomonotonicity of Browder–Hess type.

Perturbation of simple wave

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We consider the equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + \Omega^2 \sin\phi\cos\phi + \omega^2\,\sin\phi + \alpha\frac{\partial\phi}{\partial t} = 0, \tag{1}$$

which determines the dynamics of the domain bounds in a weak ferromagnet [1].

The simple (or traveling) wave solutions $\phi = \Phi_0(\varkappa x - \nu\, t)$, $\varkappa, \nu = \text{const}$, are determined by the ordinary differential equation

$$[\nu^2 - c^2\varkappa^2]\frac{d^2\Phi}{ds^2} + \Omega^2 \sin\phi\cos\Phi + \omega^2\,\sin\Phi - \alpha\,\nu\frac{d\Phi}{ds} = 0. \tag{2}$$

The simple wave under the boundary condition

$$\Phi(s) \to 0 \;\text{ as } s \to -\infty, \;\; \Phi(s) \to \pi \;\text{ as } s \to +\infty. \tag{3}$$

corresponds to the domain bound. Such solution $\Phi_0(\varkappa x - \nu\, t)$ exists if the coefficients of the equation are constant.

We study the equation with slowly varying coefficients $c^2, \Omega^2, \omega^2, \alpha$, which depend on the slow time $\tau = \varepsilon t$, [2]. Here $0 < \varepsilon \ll 1$ is a small parameter. For problem (1), (3) with the initial data corresponding to an unperturbed simple wave, we construct the asymptotic solution

$$\phi(x,t;\varepsilon) = \Phi(s;\xi,\tau)[1 + \mathcal{O}(\varepsilon)], \quad as\ \varepsilon \to 0,\ for\ t \in [0, \mathcal{O}(\varepsilon^{-1})].$$

The leading term depends on the fast variable $s = \varepsilon^{-1}S(\xi,\tau)$ and on the slow variables $\xi = \varepsilon\, x$, $\tau = \varepsilon\, t$. The function $\Phi(s;\xi,\tau)$ is a solution to equations (2), (3) under $\varkappa = S_\xi$, $\nu = S_\tau$. The phase function $S(\xi,\tau)$ is formed by two solutions of the Hamilton-Jacobi equations

$$((S_\tau^\pm)^2 - c^2(S_\xi^\pm)^2)\lambda_\pm^2 \mp \alpha S_\tau^\pm\lambda_\pm - \delta^\pm = 0,\ \delta^\pm = \pm\omega^2 - \Omega^2. \tag{4}$$

Here the constants $\lambda_\pm > 0$ are taken from the unperturbed solution in asymptotics at infinity,

$$\Phi_0(s) = \exp(\lambda_\pm s)[c_\pm + \mathcal{O}\big(\exp(\lambda_\pm s)\big)],\ s \to \pm\infty,\ c_\pm = \text{const} \neq 0.$$

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On the existence of almost periodic solutions for systems governed by differential equation and sweeping process

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Let H be a real separable Hilbert space. We consider the existence of almost periodic solutions for systems governed by a differential equation and a sweeping process of the following form

$$x'(t) = Ax(t) + f(t, x(t), u(t)),$$

$$-u'(t) \in N_{C(t)}(u(t)) + g(t, x(t), u(t)),$$

where $A : D(A) \subset H \to H$ is a linear monotone operator generating a bounded C_0-semigroup in H, $f : \mathbb{R} \times H \times H \to H$ and $g : \mathbb{R} \times H \times H \to H$ are continuous monotone operators, and N_C is a normal cone defined for a closed convex set $C \subset H$.

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On certain new classes of integral operators in complex analysis

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We will discuss two classes of operators in complex analysis, which, despite similar definitions, possess quite different properties.

In recent papers, some classes of operators in complex analysis have been studied, containing, among others, classical operators of mathematical physics. For example, in [1] we introduce and study the class of Hausdorff–Berezin operators on the unit disc based on the Haar measure. This class includes the Berezin transform itself, as well as some other classical operators, such as the invariant Green potential. We discuss certain algebraic properties of these operators, obtain conditions for their boundedness, and discuss questions of approximation of functions by some constructions in the form of Hausdorff–Berezin operators (see also [2]).

A related new class of operators called the Hausdoff–Zhu operators by the authors, also appears naturally in problems of spectral representations (we refer to [3]). Conditions of boundedness, compactness, and nuclearity of this operators are given. A special attention is paid to particular but important cases of analytic symbols and radial symbols. It is shown in particular that Hausdorff–Zhu operators with analytic symbols are at most two-dimensional and their spectra are computed.

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Nonlocal dynamics of a model of coupled oscillators with large parameter and delay

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We study nonlocal dynamics of the following model of N ($N \geqslant 2$) identical oscillators with nonlinear feedback:

$$\begin{gathered} \dot{u}_j + u_j = \lambda F(u_j(t-T)) + \gamma(u_{j-1} - 2u_j + u_{j+1}), \quad j = 1, \dots, N, \\ u_0 = u_N, \quad u_{N+1} = u_1. \end{gathered} \tag{1}$$

Here u_j are real functions, F is a piecewise continuous function such that $F(x) \equiv 0$ for $|x| \geqslant p$ (where $p > 0$ is a constant), the delay T is a positive constant, the nonzero coupling parameter γ satisfies the inequality $\gamma > -\frac{1}{4}$ (this condition is necessary for system (1) to be dissipative).

The key assumption of the study is that the positive parameter λ in system (1) is large enough ($\lambda \gg 1$).

We are interested in nonlocal dynamics of model (1). With the help of a special method of large parameter, we construct asymptotics of the relaxation modes of this model.

We select a special set S of initial conditions in the phase space $C([-T,0];\mathbb{R}^N)$ and construct the asymptotics of all solutions of system (1) with initial conditions from this set. By asymptotics, we get a finite-dimensional Poincare mapping refining the parameter values involved in the formulas for the solution asymptotics.

It was proved for positive values of the parameter γ that all the oscillators are synchronized starting from some point in time. For $-\frac{1}{4} < \gamma < 0$, in the case of an even number of oscillators, it was shown that all oscillators with even indices and all oscillators with odd indices are synchronized separately. It was shown for the case of an odd number of oscillators that all N parameters indicating the Poincare mapping are close to the same two-dimensional subspace of the original N-dimensional space with all iterations of the mapping.

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Bifurcations in second-order differential equations with delay

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Consider the singular perturbed second-order differential equation with delay

$$\varepsilon^2\ddot{x} + \sigma(\varepsilon)\dot{x} + k(\varepsilon)x = f(x(t-\tau(x)), \dot{x}(t-\tau(x)), \varepsilon), \quad 0 < \varepsilon \ll 1. \tag{1}$$

Here $x \in \mathbb{R}$, f and $\tau \geqslant 0$ are nonlinear sufficiently smooth functions. Let $f(0,0,\varepsilon) = 0$, so $x \equiv 0$ is an equilibrium. Consider some small (ε independent) neighbourhood of zero. Let $\tau(x)$ be bounded by M for all x from this neighbourhood: $0 \leqslant \tau(x) \leqslant M$.

We study dynamics of (1) in the small neighbourhood of the zero equilibrium state in the phase space $C^1_{[-M,0]}$ for sufficiently small ε.

Note that equation (1) may be obtained from the equation with large delay

$$\ddot{x} + a\dot{x} + bx = g(x(t-T\tau(x)), \dot{x}(t-T\tau(x))) \quad (T \gg 1).$$

In this case, $\varepsilon = T^{-2}$, $\sigma(\varepsilon) = \sqrt{\varepsilon}a$, $k(\varepsilon) = b$, and $f(x,y,\varepsilon) = g(x, \sqrt{\varepsilon}y)$.

In the case of a large constant delay $\tau(x) \equiv T$, this problem was discussed in [1], and in the case $f(x,y,\varepsilon) = f(x,\varepsilon)$ in [2]. In the case of a large state-dependent delay, results can be found in [3].

In the situation where all points of the spectrum of the linear part of (1) have negative real parts (separated from zero) for small enough ε, the dynamics is trivial:

all solutions tend to zero. If for small enough ε at least one point of the spectrum has positive real part, then the dynamics become nonlocal: there are no stable solutions in the neighbourhood of zero. In all other situations there exist points of the spectrum located arbitrarily close to the imaginary axis. The situations where the number of such points is infinitely large are of greatest interest. We say that such critical cases have infinite dimension.

In the critical cases, special nonlinear equations are constructed, namely, normal and quasinormal forms, that either do not depend on a small parameter, or depend on it regularly. Their solutions determine the main parts of the asymptotic expansion of solutions of (1).

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Computation and tracing of stationary solutions of Marchuk–Petrov model

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Mathematical models with time delays are widely used to analyze the mechanisms of the virus infections dynamics [1, 2]. These models are usually calibrated according to acute forms of the virus infections dynamics. At the same time, they can also be used to study chronic forms that are more difficult to treat. However, this potential of the models has not been practically realized yet, in particular, due to the fact that finding all possible stationary solutions of a given model that correspond to chronic forms turned out to be a nontrivial problem, and until recently there were no known algorithms for solving it.

This talk is devoted to the technology proposed in [3, 4] for computing all stationary solutions of a given virus infection model with fixed values of the model parameters and analysing their stability. As a development of this technology we propose to compute stationary solutions of the model using methods of computer algebra. Moreover, we propose a new method for tracing all stationary solutions along a given parameter of the considered virus infection model. This method is a modification of the method proposed in [3]. To demonstrate the new technology, we use the Marchuk–Petrov model of antiviral immune response [5] calibrated according to the dynamics of the hepatitis B virus infection. This model and its subsequent modifications developed to describe immunophysiological reactions of the body as well as mixed viral and bacterial infections are classic mathematical models used to analyze the mechanisms of viral diseases.

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The two and one-half dimensional Vlasov–Poisson system: well-posedness and stability of confined steady states

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We consider a two-component collisionless plasma described by the Vlasov–Poisson system with an external magnetic field. The setting is two and one-half dimensional meaning that the distribution functions of the ions and electrons are independent of the third space dimension. We first discuss the existence and uniqueness of global classical solutions to the corresponding initial value problem. Based on the ideas of [1] and [2], we construct confined steady states meaning that the support of their distribution functions stays away from the wall of the confinement device. Lastly, as the main part of [3], we discuss the stability of such steady states with respect to

perturbations of the initial data via the energy-Casimir method, and with respect to perturbations of the external magnetic field.

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On uniqueness of the classical solution to the Dirichlet problem for a parabolic system on the plane

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In the strip $D = \mathbb{R} \times (0, T)$, $0 < T < \infty$, the parabolic operator

$$Lu = \partial_t u - \sum_{k=0}^{2} A^{(k)}(x)\partial_x^k u,$$

is considered, where $u = (u_1, \dots, u_m)^T$ and $A^{(k)}(x) = (a_{ij}^{(k)}(x))_{i,j=1}^m$. The operator L is assumed to satisfy the following conditions:

a) the uniform parabolicity, i.e., the eigenvalues $\lambda_k(x)$ of the matrix $A^{(2)}(x)$ satisfy the inequality

$$\operatorname{Re} \lambda_k(x) \geqslant \mu > 0 \quad \forall x \in \mathbb{R}; \tag{1}$$

b) the coefficients are bounded and satisfy the Hölder condition,

$$a_{ij}^{(k)} \in C^{\alpha}(\mathbb{R}), \quad 0 < \alpha < 1. \tag{2}$$

In [1–3], the first boundary value problem for a parabolic system on the plane was considered. In particular, uniqueness of the solution in the space $C^{1,1/2}(\bar{\Omega})$ was established in domains with non-smooth lateral boundaries of the form $x = g(t)$ with a function g satisfying the condition

$$g \in C^{(1+\alpha)/2}([0, T]). \tag{3}$$

In a bounded domain Ω of the form $\Omega = \{(x,t) \in D \,|\, g_1(t) < x < g_2(t)\}$ with $g_1(t) < g_2(t)$, $0 \leqslant t \leqslant T$, we consider the first boundary value problem

$$\begin{cases} Lu &= 0 & \text{in } \Omega, \\ u(g_1(t),t) &= \psi_1(t), & 0 \leqslant t \leqslant T, \\ u(g_2(t),t) &= \psi_2(t), & 0 \leqslant t \leqslant T, \\ u(x,0) &= \varphi, & g_1(t) < x < g_2(t). \end{cases} \tag{4}$$

We prove uniqueness of the solution in the class of functions continuous up to the boundary.

Theorem. *Let conditions* (1), (2) *be satisfied for the operator L and condition* (3) *hold for functions g_1 and g_2, and let $\psi_i \in C([0,T])$, $\varphi \in C([\psi_1(0), \psi_2(0)])$, $\varphi_i(g_i(0)) = \psi_i(0)$, $i = 1,2$. Then there exists at most one classical solution $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ of the first boundary value problem* (4).

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On blow-up conditions for solutions of inequalities with the ∞-Laplacian

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Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 2$, such that $S_r \cap \Omega \neq \varnothing$ for all $r > r_0$, where S_r is a sphere in $\mathbb{R}^n$ of radius r centered at zero and $r_0 > 0$ is a real number. We study non-negative solutions of the problem

$$\Delta_\infty u \geqslant f(x,u) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where $\Delta_\infty u = \frac{1}{2}\nabla|\nabla u|^2 \nabla u$ is the infinity Laplace operator. It is assumed that $f(x,t_2) > f(x,t_1)$ for all $t_2 > t_1 > 0$ and $x \in \Omega$ and, moreover,

$$\inf_{x \in K} f(x,t) > 0$$

for any compact set $K \subset \Omega$ and real number $t > 0$. Also let there be a real number $\sigma > 1$ and non-negative measurable functions g and q such that

$$\inf_{x \in \Omega_{r,\sigma}} f(x,rt) \geqslant q(r)g(t)$$

for all $t > 0$ and $r > r_0$, where $\Omega_{r,\sigma} = \{x \in \Omega : r/\sigma < |x| < r\sigma\}$. In so doing, we assume that $g(t_2) \geqslant g(t_1) > 0$ for all $t_2 \geqslant t_1 > 0$.

Solutions of (1) are understood in the viscosity sense [1]. In the case of $\partial\Omega = \varnothing$, i.e. $\Omega = \mathbb{R}^n$, the boundary condition in (1) is valid automatically.

Theorem 1. *Let*

$$\int_1^\infty (g(t)t)^{-1/4}\,dt < \infty \quad \textit{and} \quad \int_{r_0}^\infty r^{-2/3}q^{1/3}(r)\,dr = \infty.$$

Then any non-negative solution of (1) *is identically equal to zero.*

For the problem

$$\Delta_\infty u \geqslant p(|x|)u^\lambda \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{2}$$

where $p \in C([0,\infty))$ is a positive non-increasing function, Theorem 1 implies the following result.

Theorem 2. *Let* $\lambda > 3$ *and*

$$\int_{r_0}^\infty r^{(\lambda-2)/3}p^{1/3}(r)\,dr = \infty.$$

Then any non-negative solution of (2) *is identically equal to zero.*

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On an integral equation for fractionally loaded boundary value problem of heat conduction

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In the first quadrant $Q = \{(x,t) \mid x > 0,\ t > 0\}$, we consider the boundary value problem

$$u_t - u_{xx} + \lambda \left\{ {}_cD^\beta_{0,t} u(x,t) \right\}\Big|_{x=\alpha(t)} = f(x,t), \tag{1}$$

$$u|_{t=0} = v(x); \quad u_x|_{x=0} = g(t)\ , \tag{2}$$

where λ is a complex parameter, $D_{0,t}^{\beta} u(x,\, t)$ is the Caputo derivative of order β, $2 < \beta < 3$, $\alpha(t)$ is a continuous increasing function, $\alpha(0) = 0$, or $\alpha(t)$ is a positive *const*.

The problem is studied in the class of continuous functions.

Inverting the differential part of problem (1)–(2) and introducing the notation

$$\mu\,(t) = \left\{ {}_cD_{0,t}^{\beta} u(x,t) \right\}\Big|_{x=\alpha(t)},$$

one transforms BVP (1)–(2) to the equivalent integral equation

$$\mu\,(t) - \lambda \int_0^t \mathrm{K}_{\beta}\,(t,\tau)\,\mu\,(\tau)\,d\tau = f_2\,(t)$$

with the kernel

$$K_{\beta}\,(t,\tau) = \frac{1}{\sqrt{\pi}2^{\beta-\frac{1}{2}}\,(t-\tau)^{\beta}} \exp\left(-\frac{\gamma^2\,(t)}{8t-\tau}\right) D_{2\beta-1}\left(\frac{\gamma\,(t)}{\sqrt{2\,(t-\tau)}}\right) - \frac{(\beta-1)\,(\beta-2)}{\Gamma\,(3-\beta)\,(t-\tau)^{\beta}},$$

where $D_{\nu}(z)$ is the parabolic cylinder function.

The kernel satisfies the estimate

$$|K_{\beta}\,(t,\tau)| \leqslant \frac{\Gamma\left(\beta-\frac{1}{2}\right)}{\pi 2^{4\beta}\sqrt{t-\tau}} + \frac{(\beta-1)\,(\beta-2)}{\Gamma\,(3-\beta)\,(t-\tau)^{\beta}},$$

i.e., the kernel has a non-integrable singularity, and the integral equation cannot be solved by the method of successive approximations when $2 < \beta < 3$.

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Variational problems with measurable bilateral constraints in variable domains

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Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$, $\{\Omega_s\}$ a sequence of domains in $\mathbb{R}^n$ contained in Ω, and $p > 1$. We assume that the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$

is compact, the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$, and the sequence of domains Ω_s exhausts the domain Ω.

Let $c_1, c_2 > 0$, and, for every $s \in \mathbb{N}$, let $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in Ω_s. We assume that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded and that $\int_{H_s} \mu_s \, dx \to 0$ for every sequence of measurable sets $H_s \subset \Omega_s$ with $\operatorname{meas} H_s \to 0$.

For every $s \in \mathbb{N}$, let $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s; for almost all $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$; for almost all $x \in \Omega_s$ and for all $\xi \in \mathbb{R}^n$, one has

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x, \xi) \leqslant c_2|\xi|^p + \mu_s(x).$$

For every $s \in \mathbb{N}$, define a functional $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ by

$$F_s(v) = \int_{\Omega_s} f_s(x, \nabla v) dx, \qquad v \in W^{1,p}(\Omega_s).$$

We assume that the sequence $\{F_s\}$ Γ-converges to a functional $F : W^{1,p}(\Omega) \to \mathbb{R}$.

Let $c_3, c_4 > 0$, and, for every $s \in \mathbb{N}$, let $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly lower semicontinuous functional. We assume that (a) $G_s(v) \geqslant c_3\|v\|^p_{L^p(\Omega_s)} - c_4$ for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$; (b) there exists a functional $G : W^{1,p}(\Omega) \to \mathbb{R}$ such that $G_s(v_s) \to G(v)$ for every function $v \in W^{1,p}(\Omega)$ and for every sequence $v_s \in W^{1,p}(\Omega_s)$ with the property $\|v_s - v\|_{L^p(\Omega_s)} \to 0$.

For every measurable functions $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$, we define

$$V(\varphi, \psi) = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}.$$

Similarly, for every measurable functions $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ and every $s \in \mathbb{N}$, we define

$$V_s(\varphi, \psi) = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

One of our main results is as follows.

Theorem 1. *Let $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ be measurable functions, and assume that there exist functions $\bar{\varphi}, \bar{\psi} \in W^{1,p}(\Omega)$ such that $\varphi \leqslant \bar{\varphi} < \bar{\psi} \leqslant \psi$ a.e. in Ω. For every $s \in \mathbb{N}$, let u_s be a function in $V_s(\varphi, \psi)$ minimizing the functional $F_s + G_s$ on the set $V_s(\varphi, \psi)$, and let $\{\bar{s}_k\}$ be an increasing sequence in $\mathbb{N}$. Then there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $u \in V(\varphi, \psi)$ such that the function u minimizes the functional $F + G$ on the set $V(\varphi, \psi)$, $\|u_{s_j} - u\|_{L^p(\Omega_{s_j})} \to 0$, and $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.*

For the above notions and the proof of Theorem 1, see [1].

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Comparison of numerical methods for solving the homogeneous Dirichlet problem for the Helmholtz equation in an arbitrary domain

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We study the shape-optimization problem for the cross-section of an active element in gas discharge laser physics. Our goal here is to get a better amplification factor [1] and justify the technological requirements for the alignment accuracy of the gas-discharge tube profile. This shape-optimization problem is reduced to a boundary value problem for the homogeneous Helmholtz equation with the zero Dirichlet boundary condition.

For the numerical study of such problems, either the finite difference method (FDM) or the finite element method (FEM) over the whole domain is used. However, by these methods, it is not always possible to satisfy the zero boundary conditions with required accuracy in the case of arbitrary cross-sections. Therefore, the method described in [2], which initially requires calculations only on the boundary of the domain, turns out to be attractive. In the proposed method the solution is represented in the polar coordinate system as a finite sum where each summand is the product of the Bessel function of an appropriate order depending on the radius by the trigonometric function of the polar angle, and by the weight factor. Each term of this sum satisfies the homogeneous Helmholtz equation, while the corresponding weight factors are selected to ensure the fulfillment of the boundary condition with a given accuracy.

In this talk, we compare the numerical results obtaining by all three methods for cross-sections allowing an exact solution (rectangle, circle, ellipse). Using the same number of nodes in all three methods we see that the "boundary-only" method is the fastest one for finding the solution of the Helmholtz equation and is intermediate between FDM and FEM with respect to the time of finding an amplification factor. Moreover, the "boundary-only" method requires less memory and fewer nodes to obtain the prescribed accuracy.

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Bifurcation analysis of periodic solutions to a nonlinear functional differential equation with a small parameter at the derivative

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We consider the functional differential equation

$$\varepsilon_1 \dot{x}(t) + x(t) + f(x(t-1)) = 0, \tag{1}$$

where $0 < \varepsilon_1 << 1$ and $f(x)$ is a sufficiently smooth nonlinear function. Bifurcations of periodic solutions from equilibria $x_* : \ x_* + f(x_*) = 0$ are studied (including multiples) when the parameters of the equation are varied. Stability of the equilibria x_* is determined by the the roots of the characteristic equation

$$\varepsilon_1 \lambda + 1 + f^{'}(x_*) e^{-\lambda} = 0, \quad \lambda \in \mathbb{C}. \tag{2}$$

When $|f^{'}(x_*)| < 1 (|f^{'}(x_*)| > 1)$, the equilibria are asymptotically stable (unstable). If $|f^{'}(x_*)| = 1$ and $\varepsilon_1 = 0$, then equation (2) has a countable number of roots located on the imaginary axis of the complex plane. Moreover, a countable number of resonant relations between these roots occur. Thus, for $|f^{'}(x_*)| = 1 + \varepsilon_2$ with $|\varepsilon_2| << 1$, the infinite-dimensional critical case in the stability problem for the equilibria x_* is realized. We prove in this case that the behavior of solutions of equation (1) in the vicinity of the equilibrium is described by a countable system of nonlinear ordinary differential equations depending uniformly on the parameters $\varepsilon_1, \varepsilon_2$. This system is called the normal form of equation (1) in the vicinity of the corresponding equilibrium. We propose an efficient algorithm for constructing the coefficients of the normal form. The structure of the normal form allows us to introduce a single "fast" variable and a countable number of "slow" variables. We then apply the averaging method to the corresponding system. The equilibria of the averaged system of "slow" variables determine the periodic solutions of equation (1), preserving the character of stability. Analysis of the equilibria of the averaged system of "slow" equations allows us to study the bifurcations of periodic solutions depending on the parameters $\varepsilon_1, \varepsilon_2$, to construct asymptotic formulas for them, and to study also the evolution of periodic solutions when the parameters of equation (1) are varied. We use numerical and analytical methods for this purpose. This approach was applied to study the dynamics of the Ikeda equation [1], the Mackie-Glass equation [2], and also the equation for the dynamics of meat flies population by Nicholson [3]. The possibility of bifurcation of a large number of stable periodic solutions from equilibria, i.e. bifurcation of multistability, is shown. When the bifurcation parameters are varied, the periodic solutions may pass through a series of the period doubling bifurcations and tranform into the chaotic attractors, thus forming the chaotic multistability.

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Kolmogorov's theory of turbulence and its rigorous one-dimensional analogy

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My talk is based on recent book [1] (which, in its turn, is based on my own works and those of two my former PhD students Andrei Biriuk and Alex Boritchev). Namely, I will talk about the viscous Burgers equation on the circle, perturbed by a random force. The Burgers equation with small viscosity (and with various forces) was suggested in late 1930's by Jean Burgers as a model for turbulence in fictitious one-dimensional fluid. I will explain that Sobolev norms of equation's solutions admit upper and lower estimates, asymptotically sharp as the viscosity goes to zero. This assertion allows one to obtain the results for the solutions, being rigorous analogies of the main predictions of the celebrated Kolmogorov theory of turbulence. Namely, of the Kolmogorov "1/3-law" for increments of the turbulent velocity-fields and of the Kolmogorov–Obukhov "5/3-law" for the energy spectrum of turbulence (I will explain these laws). Then I will discuss the inviscid limit for the Burgers equation and its relation with inviscid one-dimensional turbulence. The results were non-rigorously obtained by physicists in 1990's, and earlier by J. Burgers in 1948, even more heuristically.

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Estimates of Green's function of the bounded solutions problem

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Let A be a complex $n \times n$-matrix. We suppose that the spectrum $\sigma(A)$ of A does not intersect the imaginary axis. In such a case the differential equation

$$x'(t) = Ax(t) + f(t), \qquad t \in \mathbb{R},$$

has a unique *bounded on* $\mathbb{R}$ continuously differentiable solution x for any *bounded* continuous function f, and this solution possesses the representation

$$x(t) = \int_{-\infty}^{\infty} \mathcal{G}(t-s) f(s)\, ds.$$

The function $\mathcal{G}$ is called *Green's function* of the bounded solutions problem.

Let A have k eigenvalues in the right half-plane $\operatorname{Re}\mu > 0$ counted according to their multiplicities, and m eigenvalues in the open left half-plane $\operatorname{Re}\nu < 0$ counted according to their multiplicities. Thus, $k + m = n$. Let

$$\gamma^+ = \min\{\operatorname{Re}\mu : \mu \in \sigma(A),\ \operatorname{Re}\mu > 0\}, \quad \gamma^- = -\max\{\operatorname{Re}\mu : \mu \in \sigma(A),\ \operatorname{Re}\mu < 0\}.$$

We also set $\gamma = \gamma^+ + \gamma^-$.

The following estimate is analogous to the Gelfand–Shilov estimate [1, p. 68] for the initial value problem.

Theorem 1. *Green's function satisfies the estimate*

$$\|\mathcal{G}(t)\| \leqslant \eta(t)e^{-\gamma^- t} \sum_{j=0}^{m-1} \frac{|t|^{j-i}}{(j-i)!} \sum_{i=0}^{j} \binom{k+i-1}{k-1} \frac{(2\|A\|)^{k+j}}{\gamma^{k+i}}$$
$$+ \eta(-t)e^{\gamma^+ t} \sum_{j=0}^{k-1} \frac{|t|^{j-i}}{(j-i)!} \sum_{i=0}^{j} \binom{m+i-1}{m-1} \frac{(2\|A\|)^{m+j}}{\gamma^{m+i}},$$

where η is the Heaviside function.

Theorem 2. *Let, in addition, the matrix A be triangular and be represented as the sum $A = D + N$ of a diagonal matrix D and a strictly triangular matrix N. Then*

$$\|\mathcal{G}(t)\| \leqslant \left(\eta(t)e^{-\gamma^- t} + \eta(-t)e^{\gamma^+ t}\right) \sum_{k=0}^{n-1} \|N\|^k \frac{|t|^k \sqrt{\gamma|t|}\, e^{\gamma|t|/2} K_{-k-\frac{1}{2}}(\gamma|t|/2)}{\sqrt{\pi} k!},$$

where K_ν is the modified Bessel function of the second kind.

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Random motion in arbitrary direction: analytical model and simulation

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We study simulation and analytical models of a multidimensional random walk of many agents. At each step, any agent can rotate by an arbitrary angle and continue

moving in the direction selected. The analytical model gives the probability to find an agent in the circular area of radius r at a given time point. The model includes parameters that control the intensity of the walking process and the characteristics of the environment. Works on this topic are mainly concerned with the discrete case of a random walk. In this article, we consider the case of a random walk continuous in spatial coordinates. We obtain some theoretical results for the analytical model with the help of the mathematical apparatus designed to work with a generalized translation. With the help of simulation, we reveal the meaning of parameters of the analytical model.

Let agents $A = \{a_1, ..., a_s\}$ be concentrated at the origin. So each agent a_i, $i = \overline{1, s}$, has position X_0 at the time t_0. Then it starts to jump from the centre and undergoes displacements $X_1, X_2, \ldots, X_n$ at times t_1, t_2,...,t_n. The resultant at t_n is $S_n = X_0 + \sum\limits_{m=1}^{n} X_m$. The displacements are assumed to be independent, the probability density of X_m is $p_m(X_m)$, and the probability density of S_n is to be found.

Let l_m denote the length of m-th jump, $m = 1, ..., n$. Let $Pr(|S_n| \leqslant r) = Pr(|S_n| \leqslant r; l_1, l_2, ..., l_n)$ be the probability that S_n lies inside or on the circle of radius r centred at the origin O. Then the following formula for $Pr(|S_n| \leqslant r)$ is valid:

$$Pr(|S_n| \leqslant r) = \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int\limits_0^\infty J_{\frac{\nu}{2}}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}-1} dt, \qquad \nu > 0, \qquad (1)$$

where J_η is the Bessel function of the first kind [1], $j_\eta(x) = \frac{2^\eta \Gamma(\eta+1)}{x^\eta} J_\eta(x)$ is the normalized Bessel function [2], and

$$\chi(r) = \begin{cases} 1 & \text{if } r \neq |S_n|, \\ \frac{1}{2} & \text{if } r = |S_n|. \end{cases}$$

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Gromov–Hausdorff stability of global attractors and inertial manifolds

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The subject of dynamical systems concerns the evolution of systems in time. In continuous time, the systems may be modeled by ordinary differential equations or partial differential equations; in discrete time, they may be modeled by difference equations or iterated maps. The emphasis of dynamical systems is the understanding of geometrical properties of orbits and long term behavior.

First we present some dynamic properties of the hyperbolic sets of smooth dynamical systems (or flows) on compact manifolds like expansivity, shadowing property, structural stability, etc.

We then discuss some recent and ongoing works on the dynamics of flows with various expansive measures. In particular, we present a measurable version of Smale's spectral decomposition theorem for flows. More precisely, we see that if a flow ϕ on a compact metric space X is measure expansive on its chain recurrent set $CR(\phi)$ and has the shadowing property on $CR(\phi)$, then ϕ has the spectral decomposition. Moreover, we consider the Gromov–Hausdorff perturbations of dynamical systems from a measurable viewpoint.

Finally, we analyze the stability of dissipative reaction diffusion equations under perturbations of the domain and equation. More precisely, we use the Gromov–Hausdorff distances between two global attractors (or inertial manifolds) and two dynamical systems to consider the continuous dependence of the global attractors (or inertial manifolds) and the stability of the dynamical systems on their global attractors (or inertial manifolds), respectively.

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Classical hypergeometric functions and Jordan–Pochhammer systems

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It is well-known [1–3, 6] that the classic Gauss hypergeometric function is given by the power series

$$F(a,\, b,\, c\, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$$

(here $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$, $k = 1,\, 2\ldots$ are the Pochhammer symbols) and its formal multivariable Lauricella analog

$$F_D^{(n)}(a_1, a_2, \ldots, a_n,\, b,\, c\, z) = \sum_{k=0}^{\infty} \frac{(a_1)_{k_1} \cdots (a_n)_{k_n} (b)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|} k_1! \cdots k_n!} z_1^{k_1} \cdots z_n^{k_n}$$

(here $|\mathbf{k}| = k_1 + \cdots + k_n$) have the representation by the hypergeometric integrals

$$F(z,\, a\; b,\, c) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \tag{1}$$

and

$$F_D^{(n)}(z,\, a_1, a_2, \ldots, a_n\; b,\, c) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1} \prod_{i=1}^{n}(1-tz_i)^{-a_i} dt. \tag{2}$$

These integral representations enable us to show that the Gauss function and the Lauricella function have close connection with the integrable Jordan–Pochhammer systems [4–6]

$$\mathrm{d}y_i = \sum_{j=1,\, j\neq i}^{m} (\beta_i y_j(z) - \beta_j y_i(z))\, \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j}, \;\; i = 1,\, 2,\, \ldots, m. \tag{3}$$

For the Gauss function, we have $m = 2$ and for the Lauricella functions, $m = n+1$. The Gauss function and the Lauricella function are some components of basis solutions of the Jordan–Pochhammer system (3) for some parameters $(\beta_1,\, \ldots \beta_m)$. We calculate the parameters $\beta_1, \ldots, \beta_m$ via the parameters a_i, $i = 1, \ldots, n$, b, c of the Gauss or Lauricella functions and, in particular, describe all basis solutions of system (3) and the generated monodromy group of (3) in terms of modifications of the Gauss and Lauricella functions.

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Mixed boundary-value problems for elliptic differential-difference equations in a bounded domain

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Elliptic functional differential equations were studied by Ph. Hartman, G. Stampacchia, A. B. Antonevich, V. S. Rabinovich, and others. Dirichlet, Neumann, and general boundary value problems for elliptic differential-difference equations in a bounded domain were considered in [1–4], see the bibliography there. In this work, we consider mixed boundary-value problems for elliptic differential-difference equations. Such problems arise in the study of elastic deformations of three-layer plates with corrugated filler in the case where two opposite borders are rigidly fixed, and the other two are free [5].

The relationship between mixed boundary-value problems for strongly elliptic differential-difference equations and nonlocal mixed problems for strongly elliptic differential equations is established. The existence, uniqueness, and smoothness of a generalized solution to these problems are proved.

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Resonant vibration amplitude of a beam of variable length

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The resonance characteristics of viscoelastic beam with moving boundaries using the Kantorovich–Galerkin method are examined in the article. The phenomenon of resonance and steady passage through resonance are analyzed.

One-dimensional systems whose boundaries move are widely used in engineering [1]. The presence of moving boundaries causes considerable difficulties in describing such systems. Exact methods for solving such problems are limited by the wave equation and relatively simple boundary conditions. Of the approximate methods, the Kantorovich–Galerkin method described in [1] is the most efficient. However, this method can also be used in more complex cases. This method makes it possible to take into account the effect of resistance forces on the system, the viscoelastic properties of an oscillating object, and also the weak non-stationarity of the boundary conditions.

The paper considers the phenomena of steady-state resonance and passage through resonance for transverse oscillations of a beam of variable length, taking into account viscoelasticity and damping forces.

Performing transformations similar to transformations [1], an expression is obtained for the amplitude of oscillations corresponding to the n-th dynamic mode. Expressions are also obtained that describe the phenomenon of steady-state resonance and the phenomenon of passage through resonance.

The expression that determines the maximum amplitude of oscillations when passing through the resonance was numerically investigated to the maximum. The dependence of the beam oscillation amplitude on the boundary velocity, viscoelasticity, and damping forces is analyzed.

In conclusion, we note that the above results make it possible to carry out a quantitative analysis of the steady state resonance and the phenomenon of passage through the resonance for systems whose oscillations are described by the formulated problem.

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On algebraic approach in finding particular solutions of certain nonhomogeneous ODEs

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It has been long sought to represent the processes of calculus as operators. Although the problem of solving nonhomogeneous linear ordinary differential equations (ODEs) with constant coefficients is widely known (see, e.g. [1,2]), recently the author of [3] proposed a method of finding a particular solution of the differential equation in question using a matrix differential operator method.

We further develop that approach. We consider a specific function on the right-hand side of the ODE as an element of a vector space. We show that the restriction of the $\frac{d}{dx}$ operator on that space is its automorphism. Then, we use this fact to show that there is always an invertible submatrix which allows us to find a particular solution. Note that the similar method can be used to find particular solutions of *difference* equations of the same kind.

We consider the higher-order nonhomogeneous linear ordinary differential equation (ODE) with constant coefficients

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \ldots + a_n y(x) = f(x), \tag{1}$$

where $y \in C^n[a,b]$ is the unknown function, $a_1, \ldots, a_k$ are given numbers, and $f \in C^n[a,b]$ is a given function.

In the abstract, we present the most demonstrative theorem in our work.

Theorem 1. *Let $f \in L^s_{\mu,C} = \mathcal{L}[e^{\alpha x}\sin\beta x, e^{\alpha x}\cos\beta x, \ldots, x^s e^{\alpha x}\sin\beta x, x^s e^{\alpha x}\cos\beta x]$ in equation* (1) *be a quasipolynomial.*

If complex numbers $\mu = \alpha + \beta\mathbf{i}$ and $\bar{\mu}$ are roots of the corresponding homogeneous equation of multiplicity m, then we consider the particular solution $y^{(0)} \in L^{m+s}_{\mu,C} = K^m_{\mu,C} \oplus L^{m,s}_{\mu,C}$, where

$$K^m_{\mu,C} = L[e^{\alpha x}\sin\beta x, e^{\alpha x}\cos\beta x, \ldots, x^{m-1}e^{\alpha x}\sin\beta x, x^{m-1}e^{\alpha x}\cos\beta x]$$

($K_{\mu,C}$ *being the kernel of the operator on the left-hand side*) *and*

$$L^{m,s}_{\mu,C} = L[x^m e^{\alpha x}\sin\beta x, x^m e^{\alpha x}\cos\beta x, \ldots, x^{m+s}e^{\alpha x}\sin\beta x, x^{m+s}e^{\alpha x}\cos\beta x].$$

In this case, we only seek the expansion coefficients of the solution in the space $L^{m,s}_{\mu,C}$, and its expansion can be found from the system of equations $R^{m,s}_{\mu} y = f$, where $R^{m,s}_{\mu}$ is obtained from the matrix corresponding to the restriction of the left-hand-side operator on $L^{m,s}_{\mu,C}$ by crossing out the first $2m$ columns and the last $2m$ rows, and f is a vector-column obtained by expanding the right-hand side in the basis $\{x^m e^{\alpha x}\sin\beta x, x^m e^{\alpha x}\cos\beta x, \ldots, x^{m+s}e^{\alpha x}\sin\beta x, x^{m+s}e^{\alpha x}\cos\beta x\}$.

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Analytical solution for fractional advection diffusion equation with variable coefficients and source term

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In a bounded domain $\Omega = (0, L)\times(0, T), 0 < L, T < \infty$, consider the following one-dimensional fractional advection diffusion equation with a spatial variable diffusion coefficient and a source term:

$$\frac{\partial^\alpha w(s,t)}{\partial t^\alpha} = \frac{\partial^2 w(s,t)}{\partial s^2} - D_s^\beta(A(s)w(s,t)) + f(s,t), \quad 0 < \alpha, \beta < 1, \tag{1}$$

with the initial and boundary conditions

$$\begin{aligned} w(s,0) &= \phi(s), && 0 < s < L, \\ w(0,t) &= w(L,t) = 0, && 0 < t < T, \end{aligned} \tag{2}$$

where α and β are parameters describing the fractional order of the temporal and the spatial derivatives, respectively, $(A(s) > 0) \in C^1(L, 0)$ is the spatial variable diffusion coefficient and $f(s,t) \in C^2(L,0) \cap C^1(0,T)$ is the source term, $\frac{\partial^\alpha w(s,t)}{\partial t^\alpha}$ is the Caputo fractional derivative operator in the time variable, and $D_s^\beta(h(s,t))$ is the fractional Riemann–Liouville derivative operator in the spatial variable [1]. In this paper we carry out a detailed discussion and development of the method of separation of variables (the Fourier method) for the solution of fractional advection diffusion equation (1), (2), where the following theorem holds.

Theorem 1. *Suppose $f(s,t) \in C^2(L,0) \cap C^1(0,T)$, $A(s) = s^\beta > 0$, and $\phi(s) \in C(0,L)$. Then the regular solution $w(s,t) \in C^2(L,0) \cap C^1(0,T)$ of boundary value problem* (1), (2) *can be represented in the form*

$$\begin{aligned} w(s,t) = \sum_{m=1}^{\infty} \Big(\phi_m E_{\alpha,1}(-\lambda_m t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_m (t-\tau)^\alpha) f_m(\tau) d\tau\Big) * \\ \Big(s + \sum_{n=1}^{\infty} s^{2n+1} \prod_{k=1}^{n} \frac{\Gamma(2k+\beta) - \lambda_m \Gamma(2k)}{\Gamma(2k+2)}\Big). \end{aligned} \tag{3}$$

Here $E_{\alpha,\beta}$ is the Mittag-Leffler functions [2], while ϕ_m and $f_m(t)$ are the decomposition coefficients of the functions $\phi(s), f(s,t)$, respectively, in a basis of the functions $S_m(s,\lambda_m) = (s + \sum\limits_{n=1}^{\infty} s^{2n+1} \prod\limits_{k=1}^{n} \frac{\Gamma(2k+\beta)-\lambda_m\Gamma(2k)}{\Gamma(2k+2)})$.

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Role of space in an eco-epidemic predator–prey system with the effect of fear and selective predation

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A predator-prey model with disease in prey is investigated in the present study. We incorporate fear effects in susceptible prey's growth rate and the disease transmission from infected to susceptible prey. The predator population are assumed to distinguish susceptible and infected prey, and they avoid the latter to reduce fitness cost. Also, we consider that additional foods are supplied to the predator population, and the predators follow logistic growth due to that. The level of fear responsible for the reduction of the birth rate of susceptible prey is found to cause instability in the system. The rate of infection and the selectivity behavior of predators also destabilize the system. In contrast, the level of fear responsible for the eradication of the disease prevalence can restore stability of the system by evacuating persistent oscillations. Providing additional foods to predators at a lower rate induces chaotic dynamics in the system. To investigate different factors affecting the species distributions, we consider that the prey and predator populations perform active movements in the spatial directions for their biological relevance. The diffusive system is analyzed, and the numerical results show different spatial patterns exhibited by the prey and predator populations.

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Estimation of the exponent of stable solutions to functional differential equations

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The definition of the exponential stability of a linear differential equation with aftereffect generalizes the classical definition of the exponential stability of an ordinary differential equation and implies the existence of constants $N, \gamma > 0$ such that for each solution $x\colon [t_0, \infty) \to \mathbb{R}$ the estimate $|x(t)| \leqslant Ne^{-\gamma(t-t_0)}\|\varphi\|$ is valid, where φ is the initial function giving rise to the solution. For equations with aftereffect, the problem of estimating the constants N and γ relevant in application to modeling real processes, is nontrivial even for the case of scalar equations. The problem of exponential stability cannot be considered completely solved if estimates for N and γ are not specified or an algorithm for the efficient calculation of them is not found.

For the exponentially stable equation

$$\dot{x}(t) + ax(t) + \int_0^h x(t-s)\,dr(s) = f(t), \quad t \geqslant 0, \tag{1}$$

where $a \in \mathbb{R}$, $h > 0$, $r\colon [0,h] \to \mathbb{R}$ is a function of bounded variation, $r(0) = 0$, the integral is understood in the Riemann–Stieltjes sense, and f is a locally integrable function, we propose an efficient method for obtaining two-sided estimates of the fundamental solution. The method allows us to find both the exponent and the coefficient of the exponential estimate of a solution with an arbitrary accuracy. It essentially relies on the a priori assumption that the fundamental solution is positive and on complete qualitative description of its behavior.

By *the fundamental solution* of equation (1) we mean a function x_0 solving equation (1) for $f(t) \equiv 0$ and such that $x_0(0) = 1$. As is known, every solution of equation (1) can be expressed in terms of the fundamental solution.

Denote $F(\lambda) = \lambda + a + \int_0^h e^{-\lambda s}\,dr(s)$, $\lambda \in \mathbb{R}$.

Theorem 1. *Suppose the function r is nondecreasing on $[0,h]$. Then if for some real $\omega > 0$ the conditions $F(-\omega) = 0$, $F'(-\omega) > 0$ are fulfilled, then the fundamental*

solution of equation (1) *has the two-sided estimate*

$$e^{-\omega t} \leqslant x_0(t) \leqslant \frac{1}{F'(-\omega)} e^{-\omega t}.$$

Theorem 2. *Suppose the function r is nonincreasing on $[0, h]$. Then if for some real $\omega > 0$ the condition $F(-\omega) = 0$ is fulfilled, then the fundamental solution of equation* (1) *has the two-sided estimate*

$$e^{(\omega - a)h} e^{-\omega t} \leqslant x_0(t) \leqslant e^{-\omega t}.$$

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Local dynamics of a second-order equation with a delay in the derivative

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Consider a second-order differential equation with delayed feedback, which is an implementation of the modified Ikeda equation with a time delay:

$$\varepsilon \frac{d^2 y}{dt^2} + \frac{dy}{dt} + \delta y = F\left(\frac{dy}{dt}(t - \tau)\right). \tag{1}$$

Here ε and δ are small and proportional parameters $0 < \varepsilon \ll 1$, $\delta = k\varepsilon$, τ is a delay parameter, real and positive. The function F is sufficiently smooth, such that $F(0) = 0$. Thus, equation (1) has a zero equilibrium state. Let study the local dynamics in the vicinity of the equilibrium state, in the phase space $C^1_{[-1,0]}$. Note that the problem under consideration is singularly perturbed.

The paper [1] considers a similar model of an optoelectronic oscillator in which the parameter δ is not small.

The characteristic quasi-polynomial of the linearized at zero equation (1) has the form:

$$\varepsilon\lambda^2 + \lambda + k\varepsilon = \lambda\beta_1 e^{-\lambda}. \tag{2}$$

It is shown that for $|\beta_1| < 1$, the zero equilibrium state is stable, and for $|\beta_1| > 1$, it is unstable. In critical cases $\beta_1 = \pm 1$, the characteristic equation has an infinite number of roots tending to the imaginary axis at $\varepsilon \to 0$. Thus, the critical cases have infinite dimension.

To study the behavior of solutions in the case of $\beta_1 = \pm 1$, quasi-normal forms are constructed — special nonlinear parabolic equations that do not contain small parameters, the solutions of which give the main part of asymptotic solutions of equation (1) uniformly over $t \geqslant 0$ in the residual.

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Exact solution of the BVP for the Helmholtz equation in a nonconvex angle with periodic boundary data

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We consider the following model boundary value problem in a plane angle Q of magnitude $\Phi > \pi$ with a complex wave number $\omega \in \mathbb{C}^+$:

$$\begin{cases} (-\Delta - \omega^2)u(x) = 0, x \in Q, \\ u(x)\Big|_{\Gamma_1} = f_1(x), \quad u(x)\Big|_{\Gamma_2} = f_2(x). \end{cases}$$

Here Γ_l for $l = 1, 2$ are the sides of the angle Q, $f_l = e^{ik_l|x|}, k_l > 0, l = 1, 2$

Problems of this type arise in many areas of mathematical physics, for example in diffusion of a desintegrating gas. The problem differs from numerous similar problems in which the boundary data are summable functions. Using the method of complex characteristics [1], we reduce the problem to the Riemann–Hilbert problem for a Neumann data on the Riemann surface of zeros of the symbol of the Helmholtz operator. We solve this problem in quadratures and we give the solution u in the Sommerfeld type form. We find the asymptotics of the solution at the vertex and we describe its uniqueness class.

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Quantisation of free associative dynamical systems

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Traditional quantisation theories start with classical Hamiltonian systems with variables taking values in commutative algebras and then study their non-commutative deformations, such that the commutators of observables tend to the corresponding Poisson brackets as the (Planck) constant of deformation goes to zero. I propose to depart from dynamical systems defined on a free associative algebra $\mathfrak{A}$. With this approach, the quantisation problem is reduced to the problem of finding a two-sided ideal $\mathfrak{J} \subset \mathfrak{A}$ satisfying the following two conditions: the ideal $\mathfrak{J}$ has to be invariant with respect to the dynamics of the system and to define a complete set of commutation relations in the quotient algebras $\mathfrak{A}_{\mathfrak{J}} = \mathfrak{A}/\mathfrak{J}$, see [1].

To illustrate this approach, I consider the quantisation problem for the Volterra family of integrable systems. In particular, I show that odd degree symmetries of the Volterra chain admit two quantisations, one of them is the well-known quantisation of the Volterra chain, and the other one one is new and not a deformation quantisation. The periodic Volterra chain admits bi-Hamiltonian and bi-quantum structures [2]. The method of quantisation based on the concept of quantisation ideals proved to be successful for quantisation of stationary Korteveg-de–Vries hierarchies [3], the Toda chain, and non-abelian systems of two homogeneous quadratic equations.

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Asymptotics of the 1D shallow water equations in the form of running waves in a basin with variable bottom with vertical and gentle walls

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The Cauchy problem for the one-dimensional shallow water equations with variable bottom $D(x)$ and localized initial data is considered [1]. The domain under consideration is confined by a vertical wall on the right, where the Neumann conditions are set, and a movable border on the left. Asymptotics of the Carrier–Greenspan transform is used to obtain equations with fixed boundaries and small nonlinear terms, which

allows one to construct (formal) asymptotics to the initial problem [2]. Wave profile changes and its relation to the Maslov index [3] are of interest.

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On neural network trainings

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The training of a neural network is very important in machine learning [2]. It is usually done through certain optimization methods such as the gradient descent (stochastic or not [1]). There are however situations where these methods fail. This is, for instance, the case where the weights associated to the network are not real or complex numbers or belong to manifolds or even fractal sets. These problems make us to consider situations where the weights lie in a general metric space. We have an error function and the problem is therefore to find what will be referred to as a training for that error. We will consider training as just an iterative process carrying an initial weight to be close to minimize the error. In this lecture we want to formalize this idea and characterize the trainings for a given error function.

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Elliptic differential-difference equations with nonorthogonal translations in half-spaces

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In the half-space $\left\{(x,y)\middle| x\in\mathbb{R}^n, y>0\right\}$, the Dirichlet problem with summable boundary-value functions is considered for the equation

$$\sum_{j=1}^{n} u_{x_jx_j}(x,y)+u_{yy}(x,y)+\sum_{j=1}^{n} a_j u_{x_jx_j}(x+h_j,y)=0, \tag{1}$$

where

$$a_0 := \max_{j=\overline{1,n}} |a_j| < 1, \tag{2}$$

while $h_j := (h_{j1}, \ldots, h_{jn})$, $j = \overline{1,n}$, are arbitrary vectors from $\mathbb{R}^n$.

We construct Poisson-like representations of solutions of such problems, treated in the sense of generalized functions, prove the infinite smoothness of these solutions outside the boundary hyperplane, and prove that they and all their derivatives extinct as $y \to +\infty$ uniformly with respect to $x \in \mathbb{R}^n$.

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On destruction of adiabatic invariance

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We present a review of some mechanisms of destruction of adiabatic invariance in slow-fast Hamiltonian systems close to integrable ones. Adiabatic invariants are approximate first integrals of such systems. Conservation of these adiabatic invariants plays a fundamental role in many problems of plasma physics, accelerators of charged particles, waveguides. We discuss destruction of adiabatic invariance due to the passage through a separatrix, the passage through a resonance, and the change of mode of motion in systems with elastic collisions. We provide examples from charged particle dynamics. The talk is based on review papers [1] and [2]. The work was supported by the Leverhulme Thrust (RPG-2018-143).

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Asymptotics of the solution to the Cauchy problem for a singularly perturbed differential operator transport equation with small diffusion

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We construct an asymptotic solution of the Cauchy problem for singularly perturbed differential operator equation

$$\varepsilon^2(U_t + \sum_{i=1}^{N} D_i(p)U_{xi}) = L_pU + \varepsilon F(U,p) + \varepsilon^3 \sum_{i=1}^{N} B_{ij,p}U_{xi,xj}, \tag{1}$$

$$\|\bar{x}\| < \infty, t > 0, p \in \{P\}, U(\bar{x},0) = U^0(\bar{x},p,\varepsilon), \tag{2}$$

where $U = U(\bar{x},t,p)$; linear operators L, B act with respect to the variable p, $0 < \varepsilon << 1$ is a small parameter. The linear operator L_p has the simple zero eigenvalue (with an eigenfunction $h_0(p)$), the other eigenvalues have negative real parts. Under certain conditions imposed on the initial data, the coefficients D, the function F, and the operators L_p, B_p, the asymptotic expansion (AE) of the solution of problem (1)-(2) is constructed as

$$U(\bar{x},t,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i (s_i(\bar{\zeta},t,p) + p_i(\bar{\xi},\tau,p)) + R = U_N + R, \tag{3}$$

here the stretched variables $\bar{\zeta}, \bar{\xi}, \tau$ are selected in a certain way. The main term of AE (3) has the form $s_0(\bar{\zeta},t,p) = \varphi_0(\bar{\zeta},t)h_0(p)$, where $\varphi_0(\bar{\zeta},t)$ is the solution of the Cauchy problem for an equation being a generalization of the Burgers-Korteweg-de Vries equation in the case of many spatial variables,

$$\varphi_{0,t} + \sum_{i,j=1}^{N} M_{i,j}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{N}(F_{i,eff}(\varphi_0))_{\zeta_i} + \sum_{i,j,k=1}^{N} B_{ijk,eff}\varphi_{0,\zeta_i\zeta_j\zeta_k} = 0. \tag{4}$$

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Mathematical modelling of immunodominance

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Human immune response to viral infection consists of innate and adaptive immune responses. Innate immune response is mainly determined by interferon produced by infected cells and preventing (or down-regulating) further infection of uninfected cells. It is followed by the adaptive immune response provided by T and B lymphocytes. Adaptive immune response can be accompanied by immunodominance which means that there are different types of antigen-specific lymphocytes and antibodies which compete with each other, and some of them can dominate or even eliminate the others. Moreover, dominating antibodies are not necessarily most efficient for virus elimination. In this case, immunodominance can weaken the immune response. In this work we will develop a new mathematical model of immunodominance and will use it to describe some aspects of this phenomenon.

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Structural stability and limit shadowing for flows

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In this talk, we study a relationship between the structural stability and another type of limit shadowing (called L-shadowing property) for flows. In the first part, we characterize the L-shadowing flows on compact metric spaces by using the local stable and unstable sets. Moreover, we show that any L-shadowing flow on a compact metric space admits the spectral decomposition.

In the second part, we prove that any structurally stable C^1 vector field on a compact smooth manifold has the L-shadowing property, and study the C^1 interior of all L-shadowing C^1 vector fields on a compact smooth manifold (this is a joint work with Keonhee Lee).

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On the operator of translation along the trajectories of solutions of random differential inclusions

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The method of the translation operator and the method of guiding functions, whose foundations were laid by M.A. Krasnosel'skii and A.I. Perov (see [3]), were extended to the study of periodic solutions of differential inclusions (see [2] and references therein).

Let (Ω, Σ) be a complete measurable space, $\mathcal{L}$ denote the σ-algebra of the Lebesgue subsets of the interval $[0,T]$, $\mathbb{B}(\mathbb{R}^n)$ be σ-algebra of the Borel subsets of $\mathbb{R}^n$ and $Kv(\mathbb{R}^n)$ be the collection of all non-empty convex compact subsets of $\mathbb{R}^n$.

We consider the problem of the operator of translation along the trajectories of a random differential inclusion of the form:

$$x'(\omega,t) \in \mathcal{F}(\omega,t,x(t)) \quad \text{a.e. } t \in [0,T], \tag{1}$$

$$x(\omega,0) = y(\omega), \tag{2}$$

where the multimap $\mathcal{F} : \Omega \times [0,T] \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ is a random u-multimap, i.e.

($\mathcal{F}_\omega 1$) $\mathcal{F}$ is measurable with respect to the σ-algebra $\Sigma \otimes \mathcal{L} \otimes \mathbb{B}(\mathbb{R}^n)$,

($\mathcal{F}_\omega 2$) for any $\omega \in \Omega$ and a.e. $t \in [0,T]$, the multimap $\mathcal{F}(\omega,t,\cdot) : \mathbb{R}^n \to Kv(\mathbb{R}^n)$ is upper semicontinuous,

($\mathcal{F}_\omega 3$) there exists a function $c : \Omega \times [0,T] \to \mathbb{R}$ such that $c(\omega,\cdot)$ is integrable on $[0,T]$ for every $\omega \in \Omega$, $c(\cdot,t)\colon \Omega \to \mathbb{R}$ is a measurable function for a.e. $t \in [0,T]$ and $\|\mathcal{F}(\omega,t,\phi)\| := \sup\{|z| : z \in F(\omega,t,\phi)\} \leqslant c(\omega,t)(1+|\phi|)$.

Notice (see [1, Theorem 4.2]) that under the conditions above, problem (1), (2) has a random solution, i.e., there exists a function $x\colon \Omega \times [0,T] \to \mathbb{R}^n$ such that

(i) for each $\omega \in \Omega$ the function $x(\omega,\cdot)$ is absolutely continuous and satisfies relations (1), (2),

(ii) the mapping $\omega \in \Omega \to x(\omega,\cdot) \in C([0,T];\mathbb{R}^n)$ is measurable.

Let $\Delta(\omega,y) \subset C([0,T];\mathbb{R}^n)$ denote for a given $\omega \in \Omega$ the set of all solutions of problem (1), (2) with initial value $y \in \mathbb{R}^n$. Consider a continuous mapping $\xi\colon C([0,T];\mathbb{R}^n) \to \mathbb{R}^n$ defined as $\xi(x) = x(T)$.

Definition 1 (see [4])**.** A multimap $\Pi\colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined as the composition $\Pi(\omega,y) = \xi \circ \Delta(\omega,y)$, is called the multioperator of translation along the trajectories of solutions to problem (1), (2).

Theorem 1. *The multioperator of translation* Π *is a random u-multimap.*

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Improved resolvent approximations in homogenization of higher order elliptic operators

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In the space $\mathbb{R}^d$, $d\geqslant 2$, we consider the following equation of even order $2m\geqslant 4$:

$$u^\varepsilon \in H^m(\mathbb{R}^d), \quad (A_\varepsilon+1)u^\varepsilon = f, \quad f\in L^2(\mathbb{R}^d),$$
$$A_\varepsilon = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a^\varepsilon_{\alpha\beta}(x)D^\beta), \tag{1}$$

with ε-periodic coefficients $a^\varepsilon_{\alpha\beta}(x)=a_{\alpha\beta}(y)|_{y=\varepsilon^{-1}x}$, $\varepsilon>0$ is a small parameter. Here, $\alpha=(\alpha_1,\dots,\alpha_d)$ is a multiindex of length $|\alpha|=\alpha_1+\dots+\alpha_d$, $\alpha_j\geqslant 0$ are integers; $D^\alpha = D_1^{\alpha_1}\dots D_d^{\alpha_d}$, $D_i=D_{x_i}$, $i=1,\dots,d$; the coefficients $a_{\alpha\beta}(y)$ are real, measurable, 1-periodic ($Y=[-1/2,1/2)^d$ is a periodicity cell) and satisfy the conditions (**C**):

$$a_{\alpha\beta}=a_{\beta\alpha}, \quad \|a_{\alpha\beta}\|_{L^\infty(Y)}\leqslant\lambda_1, \quad \forall\alpha,\,\beta,\,|\alpha|=|\beta|=m,$$

$$\int_{\mathbb{R}^d}\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\beta\varphi D^\alpha\varphi\,dx \geqslant \lambda_0\int_{\mathbb{R}^d}\sum_{|\alpha|=m}|D^\alpha\varphi|^2\,dx \quad \forall\varphi\in C_0^\infty(\mathbb{R}^d)$$

with some positive constants λ_0 and λ_1.

It is known from long ago [1] that the family $\{A_\varepsilon\}_\varepsilon$ is G-convergent to the limit operator $\hat{A}=(-1)^m\sum_{|\alpha|=|\beta|=m} D^\alpha\hat{a}_{\alpha\beta}D^\beta$ which is of the same class (**C**) but with constant coefficients. Recently [2], it was proved that $(A_\varepsilon+1)^{-1}=(\hat{A}+1)^{-1}+O(\varepsilon^2)$ in the operator norm in $L^2(\mathbb{R}^d)$. Now we find the following approximation for the resolvent $(A_\varepsilon+1)^{-1}$ in the energy operator norm (i.e., from $L^2(\mathbb{R}^d)$ to $H^m(\mathbb{R}^d)$):

$$(A_\varepsilon+1)^{-1}=(\hat{A}+1)^{-1}+\varepsilon^m K_m(\varepsilon)+\varepsilon^{m+1}K_{m+1}(\varepsilon)+O(\varepsilon^2), \tag{2}$$

where $K_m(\varepsilon)$ and $K_{m+1}(\varepsilon)$ are certain correctors. As for the solution u^ε to (1), relation (2) means that $\|u^\varepsilon-v^\varepsilon\|_{H^m(\mathbb{R}^d)}\leqslant C\varepsilon^2\|f\|_{L^2(\mathbb{R}^d)}$, where $C=const(\lambda_0,\lambda_1,d)$ and $v^\varepsilon(x)=u(x)+\varepsilon^m\sum_\gamma N_\gamma(x/\varepsilon)\Theta^\varepsilon D^\gamma u(x)+\varepsilon^{m+1}\sum_\delta N_\delta(x/\varepsilon)\Theta^\varepsilon D^\delta u(x)$ with summation over all indices γ, $|\gamma|=m$, and δ, $|\delta|=m+1$. Here, $u=(\hat{A}+1)^{-1}f$; $N_\gamma(y)$ and $N_\delta(y)$ are

solutions to the cell problems on Y; Θ^ε is the smoothing operator needed due to lack of regularity of data in (1).

Asymptotic of type (2) was earlier [3] found under conditions (**C**) without assuming symmetry of coefficients. In this case, the asymptotic turns to be more complicated in comparison with (2).

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Reflection of functions, geometry in space and regularity of the Laplace transform

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The first theme of the article is the theorem 1. In the theorem 1 we consider a two scalar production in the linear subspace of the x_1, x_2 vectors, if $||x_1|| = ||x_2||$, and Q_1, Q_2 are the diagonals of rhombus with the x_1, x_2 sides. The first production is the usual $(x, y) = (x, y)_1$ production. The second production we can determine with help of the equality

$$(X_1, X_2)_2 = C_1R_1 + C_2R_2, \quad e_1 = x_1/||x_1||, \quad e_2 = x_2/||x_2||, \quad (x_1, x_2) \neq 0,$$

$$X_1 = C_1e_1 + C_2e_2, \quad X_2 = R_1e_1 + R_2e_2.$$

From point of the new production it is possible to suppose (see [1, theorem 1]), that the sides of rhombus are orthogonal as result of the A_1 orthogonal transformation with the A matrix: $(1/\sqrt{2}(1, 1)$ is the first line, and $(1/\sqrt{2}(1, -1)$ is the second line of the A matrix in the e_1, e_2 basis, $A^2 = E, A = A^{-1} = A^\top$, $A_1 = A_1^{-1}$.

Theorem 1. $(A_1y_1, A_1y_2) = (A_1y_1, A_1y_2)_2$ *for all the* y_1, y_2 *vectors. We can use*

$$(A_1y_1, A_1y_2)_2 = (A_1(\alpha_1e_1 + \beta_1e_2), A_1(\alpha_2e_1 + \beta_2e_2))_2 =$$

$$= \alpha_1\alpha_2(A_1e_1, A_1e_1)_2 + \beta_1\beta_2(A_1e_2, A_1e_2)_2 = J,$$

(we use $(A_1e_i, A_1e_j) = (A_1e_i, A_1e_j)_2 = 0, i \neq j$*), with help of* $(A_1e_i, A_1e_i)_2 = 1$*, if* $(e_i, e_i)_2 = 1, i = 1, 2$ *(as the result of the orthogonal transformation of the* e_1, e_2 *vectors,* $(e_1, e_2)_2 = 0$*),* $(A_1e_i, A_1e_i) = 1, i = 1, 2$ *— it is the same length 1 as on the sides of the rhombus), and*

$$J = \alpha_1\alpha_2(A_1e_1, A_1e_1) + \beta_1\beta_2(A_1e_2, A_1e_2) = (A_1y_1, A_1y_2)_1,$$

$\alpha_ie_1 + \beta_ie_2 = y_i, i = 1, 2.$

From the theorem we get

$$(A_1A_1x_1, A_1A_1x_2) = (A_1A_1x_1, A_1A_1x_2)_2 = 0, \quad A_1A_1x_1 = x_1, A_1A_1x_2 = x_2.$$

As a result of the $u(x,y) = u(-x,y) = u(x-2A) = U(x,y)$ formula (for all $x = A > 0, x = Re\, p > 0$) we get a basic fact of the second theme, if $p = x + iy = A + iy, A > 0, y \in (-\infty, \infty)$; in the formula the $U(x,y) = u(x,y)$ function is the complex part of the regular $f(p) = u(x,y) + iv(x,y)$ function, if $Re\, f(p) = f(p), p \in (-i\infty, i\infty)$, or $F(p) = \overline{f(p)}$, by definition, $F(p) = f(p-2A)$ for all $A = x = Re\, p > 0$ ($F(p)$ is the "moving field"). The $F(p) = f(p)$ equality is possible for some regular functions (from some point of view). The principle possibility of the $z = f(p)$ representation of the points of the plane we obtain from the next reasoning: the $z = f(p)$ equations of the representation of the points of the plane (in the primary system of co-ordinates with the $(0,0)$ center) for the second system of co-ordinate with the $(0,a)$ centers is equal to $z = g(w)$, $a > 0$ (for the both systems co-ordinates the z, w variables are the complex variables); if p and w is the same one point on plane, we obtain $r - w = a$, and $z = f(p) = g(p-a)$, but it is impossible: in the expression the p, w points are not the same points (in the situation for the primary representation-function we obtain the new $z = g(p-a)$ equation with $p = w$ as the same point). But we have just assumed, that p, w is the same one point in the $w = p - a$ equality. The same result we obtain from the fact: the same equation is the result of change of the direction of the OY axis and the rotation around the OX axis of the $F(p)$ field with help of regularity of the rotation.

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Fractional differential equations and their approximations

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This talk is devoted to the well-posedness and approximation for nonhomogeneous fractional differential equations

$$D^\alpha u(t) = Au(t) + f(t), \quad u(0) = u^0$$

in a Banach space E, where $0 < \alpha < 1$ and the operator A generates analytic α-resolution family. First of all, the same way as for the integer derivative, we get a necessary and sufficient condition for the well-posedness of the nonhomogeneous [2] fractional Cauchy problem in the Hölder spaces $C_0^\beta([0,T];E)$. Second, by using implicit and explicit difference schemes, we deal with the full discretization of the solutions of nonhomogeneous fractional differential equations [1] in time variables, and get the stability of the implicit difference scheme and the explicit difference scheme.

Finally, the order of convergence was obtained in the space $C([0,T];E)$ when approximating nonhomogeneous fractional differential equations by the implicit and the explicit difference schemes. For the uniform grid, it is $O(\tau^\alpha)$.

We also consider a non-uniform grid. The stability and accuracy estimates for a proposed finite difference scheme [3, 4] are obtained.

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Solvability of a linear degenerate equation with the Dzhrbashyan–Nersesyan derivative

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Let $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, and D_M be the domain of an operator M. Define the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})\}$ of an operator M, and denote $R^L_\mu(M) := (\mu L - M)^{-1}L$, $L^L_\mu := L(\mu L - M)^{-1}$. An operator M is called (L,σ)-*bounded* if $\exists a > 0 : \ \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M))$.

The operators $P = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{X})$, $Q = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{Y})$ are projections. Put $\mathcal{X}^0 = \ker P$, $\mathcal{X}^1 = \mathrm{im}P$, $\mathcal{Y}^0 = \ker Q$, $\mathcal{Y}^1 = \mathrm{im}Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathcal{X}^k$ ($D_{M_k} = D_M \cap \mathcal{X}^k$), $k = 0,1$. Denote $G := M_0^{-1}L_0$. For $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the operator M is called (L,p)-bounded if it is (L,σ)-bounded, $G^p \neq 0$, and $G^{p+1} = 0$.

Consider the initial value problem

$$D^{\sigma_k}x(0) = x_k, \quad k = 0,1,\ldots,n-1, \tag{1}$$

for the linear inhomogeneous fractional order equation

$$D^{\sigma_n}Lx(t) = Mx(t) + g(t), \tag{2}$$

in which D^{σ_n} is the Dzhrbashyan—Nersesyan fractional derivative defined according to a set of numbers $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $0 < \alpha_k \leqslant 1$, $k = 0, 1, \ldots, n$, $g \in C([0,T]; \mathcal{Y})$.

A solution to problem (1), (2) is a function $x : (0,T] \to D_M$ such that $Mx \in C((0,T]; \mathcal{Y})$, $D^{\sigma_k} x \in C([0,T]; \mathcal{X})$, $k = 0, 1, \ldots, n-1$, $D^{\sigma_n} Lx \in C((0,T]; \mathcal{X})$, equality (2) is valid for all $t \in (0,T]$, and conditions (1) hold.

Theorem 1. *Let the operator M be (L,p)-bounded, $0 < \alpha_k \leqslant 1$, $k = 0, 1 \ldots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $g \in C([0,T]; \mathcal{Y})$, $(D^{\sigma_n} G)^l M_0^{-1}(I-Q)g \in C((0,T]; \mathcal{X})$, $l = 0, 1, \ldots, p$, $D^{\sigma_k}(D^{\sigma_n} G)^l M_0^{-1}(I-Q)g \in C([0,T]; \mathcal{X})$ for $k = 0, 1, \ldots, n-1$, $l = 0, 1, \ldots, p$, $x_k \in \mathcal{X}$ satisfy the conditions $(I-P)x_k = -D^{\sigma_k} \sum\limits_{l=0}^{p} (D^{\sigma_n} G)^l M_0^{-1}(I-Q)g(t)|_{t=0}$, $k = 0, 1, \ldots n-1$. Then there exists a unique solution to problem (1), (2), and this solution can be written in the form*

$$x(t) = \sum_{k=0}^{n-1} t^{\sigma_k} E_{\sigma_n, \sigma_k+1}(t^{\sigma_n} L_1^{-1} M) P x_k +$$

$$+ \int\limits_0^t (t-s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t-s)^{\sigma_n} L_1^{-1} M) L_1^{-1} Q g(s) ds - \sum_{l=0}^{p} (D^{\sigma_n} G)^l M_0^{-1}(I-Q)g(t).$$

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Geometric flows and shape optimization

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The talk is devoted to the applications of the theory of geometric flows to shape optimization problems. As an example, we consider the problem of identifying the shape of elastic inclusion in $2D$ membrane. Assume that elastic material occupies a bounded region $\Omega \subset \mathbb{R}^2$ with the smooth boundary $\partial\Omega$. The elastic inclusion occupies the subdomain $\Omega_i \subset \Omega$ with the boundary Γ. We also assume that the curve Γ admits the parametrization $x = \gamma(\theta)$, $\theta \in \mathbb{R}^1/2\pi\mathbb{Z}^1$. The equilibrium equations for the displacement field $u : \Omega \to \mathbb{R}^1$ have the form

$$\text{div } (\mathbf{a}\nabla u) = 0 \ \text{ in } \ \Omega, \quad u = f \in W^{1/2}(\partial\Omega) \ \text{ on } \ \partial\Omega. \tag{1}$$

Here the function $\mathbf{a}$ is defined by the equalities $\mathbf{a} = 1$ in $\Omega \setminus \Omega_i$, $\mathbf{a} = a_0$ in Ω_i, $a_0 = \text{const} > 0$, and f is a given displacement field. The problem of the identification of the inclusion is formulated as follows. For a given function $g : \partial\Omega \to \mathbb{R}^1$, it is

necessary to find Γ such that the solution to problem (1) satisfies the extra boundary condition

$$\nabla u \cdot \mathbf{n} = g \quad \text{on} \quad \partial\Omega. \tag{2}$$

This problem is ill-posed. Its approximate solution can be found by solving the variational problem $\min\limits_{\Gamma\in\mathcal{A}} \mathbf{J}(\Gamma)$, where $\mathcal{A}$ is some class of admissible curves and the Cohn–Vogelius cost functional J is defined by the equality

$$J(\Gamma) = \int_\Omega \mathbf{a}\nabla(v-w)\cdot\nabla(v-w)\,dx. \tag{3}$$

Here $v, w : \Omega$ satisfy equation (1) and the boundary conditions $v = f$, $\mathbf{a}\nabla w\cdot\mathbf{n} = g$ on $\partial\Omega$. The question is the construction of a robust algorithm for the numerical study of the identification problem. The standard approach is to use the steepest descent method based on the shape calculus, see [1,4] and references therein. The main idea is to study the shape derivatives dJ of J. Without loss of generality we may restrict our considerations by the class of twice differentiable immersions $\gamma : S^1 \to \mathbb{R}^2$ with $\Gamma = \gamma(S^1)$. Choose an arbitrary vector field $\varphi : S^1 \to \mathbb{R}^2$ and consider the immersions $\gamma_t(\theta) = \gamma(\theta) + t\varphi(\theta)$, $t \in (-1, 1)$. The shape derivative $dJ = \Phi\mathbf{n} : \Gamma \to \mathbb{R}^1$ is defined by the equality

$$\partial_t\, J(\gamma_t)\Big|_{t=0} = \int\int_0^{2\pi} \Phi(\theta)\,\varphi\cdot\mathbf{n}\,d\theta,$$

where $\mathbf{n}$ is the inward normal to $\partial\Omega_i = \Gamma$. The shape derivative of the Cohn–Vogelius functional is given by the formula

$$dJ = \big\{2\big(\mathbf{a}\nabla v\cdot n\big[\partial_n v\big] - \mathbf{a}\nabla w\cdot n\big[\partial_n w\big]\big) - \big[\mathbf{a}\nabla v\cdot\nabla v - \mathbf{a}\nabla w\cdot\nabla w\big]\big\}|\partial_\theta\gamma|\,\mathbf{n}. \tag{4}$$

In the steepest descent method, the optimal shape $\Gamma_{opt} : x = \gamma_{opt}(\theta)$ is determined as the limit $\gamma_{opt}(\theta) = \lim\limits_{t\to\infty}\gamma(\theta, t)$, where $\gamma(\theta, t)$ is a solution to the Cauchy problem

$$\partial_t\gamma = -dJ(\gamma) \quad \gamma(\theta, 0) = \gamma_0(\theta). \tag{5}$$

However, the analysis of problem (5) meets serious difficulties. The way to cope with these difficulties is to use the regularization of the cost functional. We define the regularized cost function using the natural geometric quantities: the Euler elastic energy $\mathcal{E}_b(\gamma)$ and length $\mathcal{L}(\gamma)$,

$$\mathcal{E}_b(\Gamma) = \frac{1}{2}\int_\Gamma |\mathbf{k}^2|\,ds, \quad \mathcal{L}(\Gamma) = \int_\Gamma ds.$$

Here $\mathbf{k}(\theta) = \partial_s^2\gamma(\theta)$ is the curvature of Γ. Note that there is a growing massive of the literature devoted to the problem on long-time evolution of closed curves moving by the gradient flow of the elastic energy. In particular, we mention seminal paper [2]. In order to avoid the problem of self-intersections of Γ, we consider the modified elastic energy. By virtue of the 1-dimensional version of Li–Yau inequality, see [3], a curve Γ has no self-intersections if $\mathcal{E}_b\mathcal{L} < \mathcal{E}_b^*\mathcal{L}^*$. Here the quantities $\mathcal{E}_b^*$ and $\mathcal{L}^*$ correspond to the energy and length of the ∞-shaped Euler elastica. We define the modified elastic energy $E(\Gamma)$ by the equality

$$E(\Gamma) = \mathcal{E}_b(\Gamma) + \big(1 + H\big(\mathcal{E}_b(\Gamma)\mathcal{L}(\Gamma)\big)\big)\,\mathcal{L}(\Gamma),$$

where $H : [0, \mathcal{E}_b^* \mathcal{L}^*)$ is a smooth, nonnegative, monotone function such that $H(s) \to \infty$ as $s \to \mathcal{E}_b^* \mathcal{L}^*$. The gradient flow for the modified elastic energy is determined by the evolution operator equation

$$\partial_t \gamma = -\varepsilon dE(\Gamma) - dJ(\Gamma), \quad \gamma(\theta, 0) = \gamma_0(\theta), \tag{6}$$

where ε is a small positive parameter. We prove that Cauchy problem (6) has a unique strong global solution. We also prove that this solution converges to a critical point of E as $t \to \infty$.

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Behaviour of solutions of the traffic flow mathematical model

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This paper presents an investigation of the traffic flow mathematical model describing the movement of $N \in \mathbb{N}$ vehicles. This model is represented by the system

$$\begin{cases} \ddot{x}_n(t) & = R_n \left[a_n \left(\dfrac{v_{max,n} - \dot{x}_{n-1}(t-\tau)}{1 + e^{k_n(x_n(t) - x_{n-1}(t-\tau) + s_n)}} + \dot{x}_{n-1}(t-\tau) - \dot{x}_n(t) \right) \right] + \\ & \quad +(1 - R_n) \left[q_n \left(\dfrac{\dot{x}_n(t) \left[\dot{x}_{n-1}(t-\tau) - \dot{x}_n(t) \right]}{x_{n-1}(t-\tau) - x_n(t) - l_{n,\varepsilon}} \right) \right], \\ x_n(t) & = \lambda_n, \quad \dot{x}_n(t) = v_n, \quad \text{for } t \in [-\tau, 0], \end{cases} \tag{1}$$

of delay-differential equations, where $\Delta x_n(t, \tau) = x_{n-1}(t - \tau) - x_n(t)$ is the distance between adjacent vehicles, τ is the driver response time, $a_n > 0$ and $q_n > 0$ are the coefficients describing technical characteristics of vehicle acceleration and deceleration, respectively, $v_{max} > 0$ is the maximum desired speed, l_n is the safe distance corresponding, $l_{n,\varepsilon} = l_n - \varepsilon$ with an additive ε to prevent a vehicle from deceleration with infinite speed when $\Delta x_n(t, \tau)$ becomes close enough to l_n, $k_n > 0$ and $s_n > 0$ are driver behavior parameters, where k_n describes the smoothness of the vehicle driver movement adapting the speed to the speed of the one in front, λ_n is the initial position of the vehicles, v_n is the initial speed of the vehicles, and R_n is a relay function as follows:

$$R_n = \begin{cases} 1, & \text{if } \Delta x_n(t, \tau) > \dfrac{\dot{x}_n^2(t)}{2\mu g} + l_n, \\ 0, & \text{if } \Delta x_n(t, \tau) \leqslant \dfrac{\dot{x}_n^2(t)}{2\mu g} + l_n, \end{cases}$$

where μ is the friction coefficient and g is the acceleration of gravity. The function $R(\Delta x_n(t,\tau))$ describes the "acceleration-deceleration" switch.

The analysis of stability of a uniform driving mode was carried out for the model. This driving mode assumes that all vehicles are moving with the same speed v_{max} at a distance $\Delta c_n = c_n - c_{n-1}$ from each other, where c_n is a decreasing sequence. For any decreasing sequence c_n, there exists a solution of system (1) of the form

$$x_n(t) = c_n + v_{max}t.$$

The stability of such a solution depends on the signs of the expressions

$$d_n = -\tau v_{max} + c_n - c_{n-1} - l_{n,\varepsilon}.$$

The following theorem is valid.

Theorem. *If the inequality $d_n > 0$ holds for all n, then the uniform mode is stable. If the inequality $d_i \leqslant 0$ holds for at least one i, then the uniform mode is unstable.*

It follows from the theorem that if all the vehicles of the flow move at a fairly large distance from each other, then this driving mode is stable. Stability is lost by increasing the speed v_{max}, the driver response time τ, the safe distance between vehicles l_n, or by reducing the distance between two adjacent vehicles Δc_n.

Some remarks on a formula for Sobolev norms due to Brezis, Van Schaftingen, and Yung

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We provide answers to some questions raised in a recent work by H. Brezis, J. Van Schaftingen, and P.-L. Yung [1, 2] concerning the Gagliardo semi-norm $|u|_{W^{s,q}}$ computed at $s = 1$, when the strong L^q is replaced by weak L^q. In particular, we address generalization of the results in [1, 2] for a general domain and non-smooth functions.

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On stability of stationary solutions in mathematical models in natural sciences and humanities

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We consider the initial boundary value problem for a system of partial differential equations

$$\frac{\partial u_s}{\partial t} = \vartheta_s \Delta u_s + F_s(u), \quad x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n, \ t > 0, \tag{1}$$

$$\left(\mu_s u_s + \eta_s \frac{\partial u_s}{\partial \nu}\right)\Bigg|_{x \in \partial\Omega} = B_s(x),$$

$$\mu_s^2 + \eta_s^2 > 0, \ \mu_s \geqslant 0, \ \eta_s \geqslant 0, \ \mu_s = const, \ \eta_s = const, \tag{2}$$

$$u_s(x, 0) = u_s^0(x), \ s = 1, \dots, m, \tag{3}$$

where Ω is a bounded domain with piecewise smooth boundary $\partial\Omega$, $\overrightarrow{\nu} = \nu$ is a unit external normal vector to the boundary $\partial\Omega$ of the domain Ω, $u = (u_1(x,t), \dots, u_m(x,t))$, $\vartheta_s \geqslant 0$, $u = (u_1(x,t), \dots, u_m(x,t))$, $\vartheta_s \geqslant 0$, $s = 1, \dots, m$, Δ is the Laplace operator defined by the formula

$$\Delta v = \sum_{j=1}^{n} \partial^2 v / \partial x_j^2.$$

Let $w = w(x) = (w_1(x_1, \dots, x_n), \dots, w_m(x_1, \dots, x_n))$ be a stationary solution of initial boundary value problem (1)–(3), functions F_s be differentiable at w,

$$A_{sk} = (\partial F_s(w)/\partial z_k, +\partial F_k(w)/\partial z_s,)/2 - \delta_{ks}\vartheta_s/d^2, s, k = 1, \dots, m.$$

The negative definiteness of the quadratic form

$$\sum_{s=1}^{m} \sum_{k=1}^{m} A_{sk} \, z_k z_s \, ,$$

is a sufficient condition for the stability of the stationary solution.

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Spectral asymptotics for fourth-order differential operator

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We consider a self-adjoint fourth-order operator H acting in the Hilbert space $L^2(0,1)$ and given by

$$Hy = y^{(4)} + (py')' + qy, \quad y'(0) = y'''(0) + p(0)y'(0) = y(1) = y''(1) = 0,$$

where the coefficients p and q are real 1-periodic functions, $p,\ q \in L^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The domain of this operator is the space

$$\mathrm{Dom}(H) = \{y \in L^2(0,1) : y', y'', y''' + py' \in L^1(0,1), y^{(4)} + (py')' + qy \in L^2(0,1), \\ y'(0) = y'''(0) + p(0)y'(0) = y(1) = y''(1) = 0\}.$$

Kitavtsev, Recke, and Wagner [1] proved that the thin film equation is reduced to the spectral problem for the operator H. The main goal of our talk is to determine high energy eigenvalue asymptotics for H. Introduce the Fourier coefficients

$$f_0 = \int_0^1 f(x)\,dx, \qquad \widehat{f}_{cn} = \int_0^1 f(x)\cos\pi(2n+1)x\,dx.$$

We formulate our main result.

Theorem 1. *Let $p,\ q \in L^1(\mathbb{T})$ and let $n \in \mathbb{N}$ be large enough. Then the eigenvalues μ_n are real and have algebraic multiplicity one. Moreover,*

$$\mu_n = \big(\frac{\pi}{2} + \pi n\big)^4 + \big(\frac{\pi}{2} + \pi n\big)^2 (\widehat{p}_{cn} - p_0) + \mathcal{O}(n),$$

as $n \to +\infty$. If, in addition, p''', $q' \in L^1(\mathbb{T})$, then

$$\mu_n = \big(\frac{\pi}{2} + \pi n\big)^4 + \big(\frac{\pi}{2} + \pi n\big)^2 \big(\widehat{p}_{cn} - p_0\big) + \frac{p_0^2 - \|p\|^2}{8} + q_0 + \widehat{q}_{cn} + \mathcal{O}(n^{-2}),$$

as $n \to +\infty$.

These results are an essential step to obtain the trace formula for the operator H and the first step in solving the inverse spectral problem for this operator.

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Optimal control problems for linear fractional-order equations under different definitions of fractional integro-differential operators

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We consider linear fractional-order equations of the following type:

$$ {}_0D_t^\sigma q(t) = a(t)q(t) + b(t)u(t), \tag{1} $$

where $q(t)$ is the system state, $q(t) \in H^1(0,T]$, ${}_0D_t^\sigma$ is the left-sided fractional derivative operator, $t \in (0,T]$, $a(t)$ and $b(t)$ are some given functions. Admissible controls are assumed to be p-integrable functions, $u(t) \in L_p(0,T]$, $p > 1$. We will discuss several cases depending whether the fractional derivative is understood in the sense of Caputo–Fabrizio, Atangana–Baleanu, Erdélyi–Kober or Katugampola. In the last two cases, index σ is composite and consists of several indices. In the first two cases, we consider $\sigma \in (0,1]$.

Let us formulate two optimal control problems (OCP A and B) for system (1). Find a control function $u(t)$, $t \in [0,T]$, such that system (1) passes from a given initial state to the assigned final state and herewith: 1) the norm of a control function is minimal with the assigned control time T (OCP A), 2) the control time T is minimal provided $\|\vec{u}\| \leqslant l$, $l > 0$, where l is the assigned constant (OCP B).

The OCP stated above can be reduced to the classical l-problem of moments [1].

Theorem 1. *Let system* (1) *be given with fractional derivative operator meant in the sense of Caputo–Fabrizio or Atangana–Baleanu. Then the l-problem of moments for system* (1) *is well-posed*

1. *at any $\sigma \in (0,1]$ in the case of Caputo–Fabrizio;*
2. *for $\sigma > 1/p$ in the case of Atangana–Baleanu.*

Analogous results were obtained for Erdélyi–Kober operators recently [2].

We construct some explicit solutions of the l-problem of moments and analyze the control function norm behaviour dependng on the derivative index for different kinds of the fractional-order derivative. Also, we analyze boundary and optimal trajectories of these systems.

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About integral invariants of multidimensional differential systems

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The theory of integral invariants (both relative and absolute) was introduced by H. Poincaré in his pioneering work [1] and later was extended by E. Cartan [2]. At the same time, important specific examples of integral invariants were known earlier (see, e.g., the theorems of W. Thomson and H. L. F. Helmholtz from hydrodynamics on the conservation of circulation and vortex flow [3, pp. 105–111]. The current state of the theory of integral invariants is given by academician V. V. Kozlov in [4].

In this talk, we present the theory of integral invariants of the first and the total orders for the system of exact differential equations (or the Pfaff system)

$$dx_i = \sum_{j=1}^{m} X_{ij}(t,x)\,dt_j\,, \quad i=1,\dots,n, \quad X_{ij}\in C^1(D), \quad D = T\times\mathcal{X}\subset\mathbb{R}^{m+n}. \tag{1}$$

The following result is proved in [5].

Theorem 1. *System* (1) *admits the first order absolute integral invariant*

$$I = \int\limits_L \sum_{i=1}^{n} J_i(t,x)\,\delta x_i\,, \quad J_i\in C^1(D'),\ D'\subset D,\ i=1,\dots,n,$$

if and only if the following identities hold

$$\mathfrak{X}_j\,J_i(t,x)+\sum_{k=1}^{n} J_k(t,x)\,\partial_{x_i}X_{kj}(t,x)=0 \quad \forall(t,x)\in D', \quad j=1,\dots,m, \quad i=1,\dots,n,$$

where L is an arbitrary smooth curve in the domain $\mathcal{X}'\subset\mathcal{X}\subset\mathbb{R}^n$, and $\mathfrak{X}_j$ are the linear differential operators $\mathfrak{X}_j(t,x)=\partial_{t_j}+\sum\limits_{i=1}^{n} X_{ij}(t,x)\,\partial_{x_i}$ $\forall(t,x)\in D,\ j=1,\dots,m$.

All obtained results were concretized for the multidimensional Hamiltonian systems

$$dq_i = \sum_{j=1}^{m}\partial_{p_i}H_j(t,q,p)\,dt_j\,, \quad dp_i = -\sum_{j=1}^{m}\partial_{q_i}H_j(t,q,p)\,dt_j\,, \quad i=1,\dots,n,$$

where the Hamiltonians H_j belong to $C^2(G)$, $j=1,\dots,m$, and G is a domain in $\mathbb{R}^{m+2n}$.

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Semigroup approach to studying Volterra integro-differential equations arising in viscoelasticity theory

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Abstract Volterra integro-differential equations with integral operator kernels representable by Stieltjes integrals are studied. These integro-differential equations can be realized as partial integro-differential equations arising in the theory of viscoelasticity and the theory of heat propagation in media with memory and have many of other important applications.The approach is based on the study of one-parameter semigroups for linear evolution equations. Results on the existence of a strongly continuous contraction semigroup generated by a Volterra integro-differential equation with operator coefficients in a Hilbert space are stated. The statement of the corresponding Cauchy problem for a first-order differential equation is given, and a theorem on the well-posedness of this problem is stated. The properties of the generator of the semigroup and the properties of the operator function associated with it (the symbol of the original integro-differential equation) are studied (see [1, 2]).

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Bound rates for convergence of FEM in the problem of wavefront reconstruction from its slope measurements with fractional order stabilizer

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The problem of wavefront reconstruction from its slopes arises in adaptive optics and is associated with determining the phase of a light wave based on the data of the Shack-Hartmann sensor [1]. In [2, 3], a family of new wavefront reconstruction

methods was proposed, which are based on the variational approach in combination with the finite-element method (FEM) and bilinear splines. In [3], to improve the spatial-frequency characteristic of the method, a stabilizer with the second order difference derivatives was used with a fractional power, the parameters of which were also chosen based on the variational approach.

The report discusses the accuracy bounds of the considered method. Under the condition that the grid step is consistent with the parameters of the fractional order stabilizer, bounds are obtained for the accuracy of FEM under natural requirements for the smoothness of the wavefront from anisotropic Sobolev spaces with integer smoothness indices. Based on the choice of special anisotropic spaces with fractional smoothness orders and the use of corresponding interpolation methods for the FEM error operator, scales of bounds for the accuracy of the method are obtained that are consistent with the fractional smoothness of slopes.

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Functional differential equations with dilation and symmetry

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Basics of the theory of boundary-value problems for elliptic functional differential equations were laid down by A. L. Skubachevskii [1], where an extensive overview of the literature and applications to problems of physics, mechanics, and control theory can be found. Elliptic functional differential equations with compression and dilation of independent variables were considered in [2, 3]. They have a notable one-dimensional prototype, the *pantograph equation* $\dot{y} = ay(\lambda t)+by(t)$, emerging in various fields such as astrophysics, engineering, and biology.

The talk is devoted to the Dirichlet problem in a plane bounded domain for a strongly elliptic second-order functional differential equation containing arguments' transformations of the form $x \mapsto x/p$ $(p > 0)$ and $x \mapsto -x$ in the highest derivatives. The study of the solvability of the problem is based on the Gårding-type inequality for which necessary and sufficient conditions are obtained in algebraic form. We combine the Gelfand theory of commutative Banach algebras with the methods for strongly elliptic systems of differential equations.

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Justification of models of plates containing hard thin inclusions inside

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The equilibrium problem for the Kirchhoff–Love plate containing a nonhomogeneous inclusion is considered. It is assumed that the elastic properties of the inclusion depend on a small parameter characterizing width of the inclusion ε as ε^N with $N < 1$. The problem is formulated as a variational one, namely, as the problrm of minimization of the energy functional over the set of all admissible deflections in the Sobolev space H^2. This implies that the deflection function is a solution of the boundary value problem for the biharmonic operator (pure bending, see, e.g., [1, 2]).

The aim of the present work is to justify the passage to the limit as $\varepsilon \to 0$. To do this, we apply a method that was originally introduced in [3, 4] for problems of gluing plates. The method is based on variational properties of the solution to the corresponding minimization problem and allows us to find limit problems for all $N < 1$ simultaneously. It is shown that there exist two types of hard inclusions depending on N: thin rigid inclusion $(N < -1)$ and thin elastic inclusion $(N = -1)$. In the case $N \in (-1, 1)$, the influence of the inhomogeneity disappears in the limit. We get limit problems in a variational form, which is convenient, for example, for numerical analysis by the finite element method.

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Estimates of total bandwidth for Schrödinger operators on periodic graphs

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We consider Schrödinger operators with periodic potentials on periodic discrete graphs. Their spectrum consists of a finite number of bands. We obtain two-sided estimates of the total bandwidth for the Schrödinger operators in terms of geometric parameters of the graph and the potentials. In particular, we show that these estimates are sharp. It means that these estimates become identities for specific graphs and potentials. The proof is based on the Floquet theory and trace formulas for fiber operators. The traces are expressed as finite Fourier series of the quasimomentum with coefficients depending on the potentials and cycles of the quotient graph from some specific cycle sets. In order to obtain our results we estimate these Fourier coefficients in terms of geometric parameters of the graph and the potentials. The talk is based on joint work with Evgeny Korotyaev from St. Petersburg State University [1].

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Nonlinear Schrödinger equation with delay and its regularization

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The report is devoted to the properties of the initial-boundary value problem for a nonlinear Schödinger equation containing terms with time delay. The conditions for the existence of a global solution are obtained, as well as the conditions for the gradient blow up phenomenon. The relation between the gradient blow up with self-focusing and the destruction of pure quantum state is described (see [1]). The procedure of continuation of the solution through the blow up by the curve in the set of general quantum states is defined similar to [2].

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Eta-invariants for G-operators

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Eta-invariants of invertible parameter-dependent pseudodifferential operators were introduced by Melrose. These invariants are a regularization of the classical winding number. They are also a generalization of the Atiyah–Patodi–Singer eta-invariant of elliptic self-adjoint operators.

We extend the definition of eta-invariants to a wide class of nonlocal operators. Let us describe our class of nonlocal parameter-dependent operators. First, let a discrete group G act isometrically on a manifold X and consider the induced representation of G on functions on X by shift operators $(T_g u)(x) = u(g^{-1}x)$. Second, we allow our operators with parameter to have periodic coefficients. Thus, we consider the following class of G-operators with parameter:

$$D(p) = \sum_{(g,k)\in G\times\mathbb{Z}} D_{gk}(x, -i\partial/\partial x, p) T_g e^{ikp} : C^\infty(X) \longrightarrow C^\infty(X), \tag{1}$$

where $D_{gk}(x, -i\partial/\partial x, p)$ is a pseudodifferential operator with parameter p (e.g., see [1]). Families of the form (1) arise when studying elliptic theory for G-operators on manifolds with cylindrical ends and on manifolds with isolated singularities. The notion of ellipticity for such families is known. As in the classical case, ellipticity implies the Fredholm property for all $p \in \mathbb{R}$ and invertibility for large p.

Given an invertible elliptic family (1), we define the eta-invariant for it, obtain its main properties (including the logarithmic property). In particular, we obtain a formula for the variation of the eta-invariant. The variation is expressed as a sum of contributions of the components of the complete symbol of the operator integrated over the cotangent bundles of the fixed point sets of the group action. The latter formula is similar to the formula for the Wodzicki residue in the case of operators without parameter. Our results in the special case of trivial G were published in [2].

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On the nonlocal integral boundary value problem for fractional differential equations

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Fractional calculus has been used to mathematically model many complicated natural biological, physical, or industrial systems. The theory of integration and differentiation of arbitrary real or complex order has attracted the attention of many mathematicians and applied scientists in various fields of science and technology since the fact that hereditary properties and memory effects of numerous real-world processes could be explained by integrals and derivatives of non-integer or fractional order. For more details on fractional differential equations and their applications, we refer the reader to [2, 4–7] and references contained therein.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics and cellular systems. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, see for instance [1, 3, 8] and the references therein.

In this work, we are concerned with the existence of solutions to the nonlocal integral boundary value problem for fractional differential equations of Caputo type. In our investigation we rely on the fixed point theory and the properties of fractional derivative and integral.

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Short-wave asymptotics for evolutionary equations with abruptly varying coefficients

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We study the Cauchy problems for hyperbolic equations with coefficients depending singularly on a small parameter (weak limits of the coefficients are either discontinuous functions or delta-type distributions). We study short-wave asymptotics in these problems, the main attention being paid to the behavior of geometric objects (Lagrangian surfaces of complex vector bundles) associated with such solutions. In particular, we describe rearrangements of these geometric objects near the supports of singularities of the coefficients.

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Minimizers of a variational problem for nematic liquid crystals with variable degree of orientation in two dimensions

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We study the asymptotic behavior as $k \to \infty$ of the minimizers of the energy

$$G_k(u) = \int_\Omega \Big((k-1)|\nabla |u||^2 + |\nabla u|^2 \Big),$$

over the class of maps $u \in H^1(\Omega, \mathbb{R}^2)$ satisfying the boundary condition $u = g$ on $\partial\Omega$, where Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$ and $g : \partial\Omega \to S^1$. The motivation comes from a simplified version of the Ericksen model for nematic liquid crystals. We will present similarities and differences with respect to the analog problem for the Ginzburg–Landau energy.

Generalized solution of the Hamilton–Jacobi equation with a three-component hamiltonian exponentially dependent on the momentum

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The following Cauchy problem for the Hamilton–Jacobi equation of evolutionary type is considered:

$$\frac{\partial u}{\partial t} + H\left(x, \frac{\partial u}{\partial x}\right) = 0, \quad t \in (0, T),\, x \in \mathbb{R}, \tag{1}$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

Here $T > 0$ is a fixed time point and $u_0(\cdot)$ is a given continuously differentiable function. Continuously differentiable functions $h(\cdot) : \mathbb{R} \to \mathbb{R}$ and $f(\cdot) : \mathbb{R} \to \mathbb{R}$ and $g(\cdot) : \mathbb{R} \to \mathbb{R}$ are also given. The functions f and g are assumed to be monotonically increasing and monotonically decreasing, respectively. Let there exist points x_* and x^* such that $f(x_*) = 0$, $g(x^*) = 0$, and the inequality $x_* < x^*$ is valid.

Problem (1), (2) is considered under the assumption that the Hamiltonian has the form

$$H(x, p) = \begin{cases} g(x)e^{-p}, & x < x_*,\, p \in \mathbb{R}, \\ h(x) + f(x)e^{p} + g(x)e^{-p}, & x_* \leqslant x \leqslant x^*,\, p \in \mathbb{R}, \\ f(x)e^{p}, & x > x^*,\, p \in \mathbb{R}. \end{cases} \tag{3}$$

A continuous generalized solution of problem (1)-(3) is determined on the base of the viscosity/minimax approach [1, 2]. To construct this solution, we apply the method of generalized characteristics [3] and solve variational problems. Sufficient conditions for the uniqueness of the generalized solution are indicated.

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Tensor invariants of dynamical systems with a finite number of degrees of freedom with dissipation

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It is well known [1–3] that a system of differential equations is fully integrable if it has a sufficient number of not only first integrals (scalar invariants) but also tensor invariants. For example, the order of the considered system can be reduced if there is an invariant form of the phase volume. For conservative systems, this fact is natural. However, for systems with attracting or repelling limit sets, not only some of the first integrals, but also the coefficients of the invariant differential forms involved have to consist of, generally speaking, transcendental (in the sense of complex analysis) functions [4–6].

For example, the problem of an n-dimensional pendulum on a generalized spherical hinge placed in a nonconservative force field leads to a system on the tangent bundle of the $(n-1)$-dimensional sphere with a special metric on it induced by an additional symmetry group. Dynamical systems describing the motion of such a pendulum have the various dissipation, and the complete list of first integrals consists of transcendental functions expressed in terms of a finite combination of elementary functions. There are also problems concerning the motion of a point over n-dimensional rotation surfaces, the Lobachevsky spaces, etc. The results obtained are especially important in the context of a nonconservative force field present in the system.

In this activity, we present tensor invariants for homogeneous dynamical systems on tangent bundles of smooth finite-dimensional manifolds. The relation between the existence of these invariants and the existence of a complete set of first integrals necessary for the integration of geodesic, potential, and dissipative systems is shown. The force fields introduced into the considered systems make them dissipative with dissipation of different signs and generalize previously considered force fields.

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Boundary optimal control and homogenization: critical case

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We consider the homogenization of an optimal control problem in which the control v is placed on the part Γ_0 of the boundary, and the spatial domain contains a thin layer of "small particles" very close to the controlled part of boundary with the Robin boundary condition posed on the boundary of those "small particles." We assume that the size of the particles and parameters involved in the Robin boundary condition are critical (and so they justify the occurrence of some "strange terms" in the homogenized problem and in the limit of the cost functional).

Large and very singular solutions to semilinear elliptic equations

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Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a bounded domain and $f(\cdot,\cdot)$ a nonnegative continuous function in $\overline{\Omega} \times \mathbb{R}^1$ such that $f(x,0) = 0 \ \forall\, x \in \overline{\Omega}$. We consider the so-called large solutions of the equation

$$-\Delta u + f(x,u) = 0 \text{ in } \Omega, \tag{1}$$

i.e. solutions $u(x)$ of (1) satisfying the boundary condition

$$\lim_{d(x)\to 0} u(x) = \infty, \quad d(x) := dist(x, \partial\Omega). \tag{2}$$

If $f = f(u)$ is a monotone function, then the existence of the large solution is connected with the well-known Keller–Osserman condition on the growth of $f(u)$ as $u \to \infty$ and with the suitable generalized K–O conditions for different classes of general nonnegative nonmonotonic nonlinearities $f(x,u)$ (S. Dumont, L. Dupaigne, O. Goubet, V. Radulescu (2007), J. Lopez-Gomez (2000), and others). More difficult is the problem of uniqueness of the large solution. In the case of smooth domain Ω and $f(u) = u^p$, $p = \frac{n+2}{n-2}$, $n > 2$, the uniqueness was firstly proved by C. Loewner, L. Nirenberg (1974). For $f(u) = u^p$, $p > 1$, the mentioned uniqueness was proved by C. Bandle and M. Marcus (1992). As to $f(x,u)$, M. Marcus and L. Veron (2003) proved the uniqueness of the large solution for C^2-smooth bounded Ω if $f(x,u) \geqslant c_0 d(x)^\alpha u^p$ $\forall\, x \in \Omega$, $\forall\, u \geqslant 0$, $p > 1$, $\alpha > 0$, $c_0 = const > 0$. Finally, the uniqueness was hypothesized in [1] for the case where $f(x,u) \geqslant c_0 \exp\left(-c_1 d(x)^{-\alpha}\right) u^p$ $\forall\, x \in \Omega$, $\forall\, u \geqslant 0$,$p > 1$, $0 < \alpha < 1$,$c_1 > 0$. We have proved the validity of this hypothesis. Moreover, we proved the uniqueness of the large solution even for a weaker condition on the degeneration of $f(x,u)$ on the boundary of Ω: $f(x,u) \geqslant c_0 h_\omega\left(d(x)\right) u^p$ $\forall\, x \in \Omega$, $p > 1$, where $h_\omega(s) = \exp\left(-s^{-1}\omega(s)\right)$, and a nondecreasing continuous function $\omega(\cdot)$ satisfies the Dini condition

$$\int_0^c s^{-1}\omega(s)ds < \infty \quad \forall\, c > 0. \tag{3}$$

Question: *is condition* (3) *also necessary for the uniqueness of the large solution*?

As was shown in [3], condition (3) is sufficient for the existence of the so-called very singular solution $u_a(x)$ of equation (1), where $a \in \partial\Omega$, i.e. a solution of (1) satisfying the boundary condition $u_0(x) = 0 \; \forall\, x \in \partial\Omega \setminus \{a\}$ with a singularity at the point a stronger than the singularity of the corresponding Poisson kernel. It is proved in [2] that condition (3) is also necessary for the existence of the very singular solution $u_a(x)$ of (1) with $a \in \partial\Omega$, which may be considered as some argument for the positive answer to the above question.

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Half-range problem in operator theory

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Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$ endowed also with an indefinite inner product $[x,y] = (Jx,y)$. Here J is the involution operator (fundamental symmetry):

$$J = J^* = P_+ - P_-, \quad J^2 = 1,$$

where $P_\pm$ are orthoprojections.

The space $(\mathcal{H}, [\,\cdot,\cdot\,]) = \mathcal{K}$ is called Krein space (or Pontrjagin space $\Pi_\varkappa$, provided that $\varkappa = \min\{\varkappa_+, \varkappa_-\} < \infty$, where $\varkappa_\pm = \mathrm{rank} P_\pm$).

A subspace $\mathcal{L}$ in $\mathcal{H}$ is said to be *nonnegative* if $[x,x] \geqslant 0$ for all $x \in \mathcal{L}$, and *maximal nonnegative* if there are no proper nonnegative extensions of $\mathcal{L}$.

A linear operator A is said to be self-adjoint (m-dissipative) in $\mathcal{K}$ if JA is self-adjoint (m-dissipative) in $\mathcal{H}$.

Problems (open even for bounded operators):

1. *Does a self-adjoint or m-dissipative operator A in $\mathcal{K}$ possess a maximal nonnegative A-invariant subspace?*

2. *If yes, is there one among these subspaces such that the spectrum of the restriction $A^+ = A|_{\mathcal{L}}$ onto this subspace lies in the closed half-plane $\mathbb{C}^-$?*

3. *If yes, does the operator A^+ generate a C_0 or holomorphic semigroup?*

These problems are closely connected with the following ones having important applications in actual problems of mathematical physics.

1. Let $L(\lambda) = \lambda^2 A + \lambda B + C$, where C is a self-adjoint (generally unbounded) operator, while symmetric operators A and B are such that $\mathcal{D}(A) \supset \mathcal{C}$, $\mathcal{D}(B) \supset \mathcal{C}$. To find conditions for the factorization

$$L(\lambda) = (\lambda - Z_1)A(\lambda - Z), \quad Z_1 = -(AZ + B)A^{-1},$$

where the spectrum of Z lies in the closed lower half-plane $\mathbb{C}^-$. What additional spectral properties of the operator Z can be discovered?

2. Let $\mathcal{L} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a self-adjoint operator in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. To find conditions under which $\mathcal{L}$ is similar to a diagonal operator in $\mathcal{H}$.

In the talk, we present old and new results concerning these problems.

Finite-time generalized synchronization between chaotic systems

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We discuss finite-time generalized synchronization between different chaotic systems. Further, chaotic and hyper-chaotic systems are synchronized up to a desired transformation matrix. Simulation for chaotic and hyper-chaotic systems has been performed. Numerical results are presented graphically.

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A matrix approach for nonlinear weakly singular integro-partial differential equations

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An efficient matrix approach for the solution of nonlinear weakly singular integro-partial differential equations (integro-PDEs) with a given initial condition is proposed. The method is bases on the operational matrices of the Legendre wavelets. By implementing the operational matrices of the Legendre wavelets, we reduce the original intero-PDE to a system of nonlinear equations. Some useful results concerning the convergence and error estimates associated to the suggested scheme are presented. Illustrative examples are provided to show the effectiveness and accuracy of the proposed numerical method.

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On solvability of parabolic differential-difference equations

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We consider the differential–difference equation with shifts of space variables

$$\partial_t u(t,x) - \sum_{1\leqslant i\leqslant n} \partial_i A_i(t,x,Ru,\nabla Ru) + A_0(t,x,Ru,\nabla Ru) = f(t,x) \quad ((x,t)\in\Omega_T) \quad (1)$$

in the cylinder $\Omega_T = (0,T)\times Q$, supplemented by the initial condition

$$u(0,x) = \varphi(x) \qquad (x\in Q) \quad (2)$$

and the boundary condition

$$u(t,x) = 0 \qquad (0<t<T,\ x\in\mathbb{R}^n\setminus Q). \quad (3)$$

Here $Q\subset\mathbb{R}^n$ is a bounded domain with boundary $\partial Q\in C^\infty$,

$$Ru(t,x) = \sum_{h\in\mathcal{M}} a_h u(t,x+h),$$

where $a_h\in\mathbb{R}$, and the set $\mathcal{M}\subset\mathbb{R}^n$ consists of a finite number of vectors $h\in\mathbb{R}^n$ with integer coordinates.

Let $\varphi \in L_2(Q)$, $f \in W_q(0,T;W_q^{-1}(Q)$, $p \in (1,\infty)$, and $1/p + 1/q = 1$. Denote $W := \{u \in L_p(0,T;\mathring{W}_p^1(Q)) : \partial_t u \in L_q(0,T;W_q^{-1}(Q))\}$. Thus, we consider the generalized solvability of parabolic equation (1) with a differential–difference operator A_R. Sufficient conditions for the operator A_R to be pseudomonotone on W or to have (X,W)-semibounded variations are obtained. We also consider conditions for A_R to be coercive or partially coercive. These conditions guarantee that problem (1)–(3) has at least one generalized solution $u \in W$.

Note that more restrictive conditions were considered in [3]. Here we use some results and some proofs from [1, 2] to obtain weaker conditions.

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A new numerical approximation of Caputo fractional derivative and its applications

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In this talk, we discuss numerical approximation of the Caputo fractional derivative of order $\alpha \in (1,2)$. We have used the cubic interpolating polynomial to develop this approximation. The approximation is second order accurate and proved to be very helpful to find the numerical solution of time-fractional diffusion wave problem. The difference scheme is second order in time and space for all α. In order to prove the effectiveness and accuracy of the scheme, a comparative study is given with the earlier existing results.

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Asymptotics of eigenvalues of large Toeplitz matrices

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Our goal is to find uniform asymptotic formulas for all of the eigenvalues of banded symmetric Toeplitz matrices of large dimension with degenerate symbol. The entries of the matrices are real, and we consider the case where the real-valued generating function is such that its first five derivatives at one endpoint of the interval equal zero. This is not the simple-loop case considered earlier. We obtain nonlinear equations for the eigenvalues. It should be noted that our equations have a more complicated structure than the equations for the simple-loop case.

Let $a(t)$ be a Lebesgue integrable function defined on the unit circle $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$. We denote by $T_n(a)$ the Toeplitz matrix $T_n(a) := (a_{j-k})_{j,k=1}^{n-1}$, where $n \in \mathbb{N}$ is a natural number, and a_l denotes the l-th coefficient of the Fourier series of the function a. Note that the Toeplitz matrix can be viewed as an operator from a finite dimensional vector space. The function $a(t)$ is called the symbol of the Toeplitz matrix (Toeplitz operator) $T_n(a)$. A model example considered in detail this report is devoted to finding asymptotic formulas for the eigenvalues of the Toeplitz matrix with the symbol $a(t) = (t - 2 + \frac{1}{t})^3$. Introduce the function $g(\varphi) = a(e^{i\varphi}) = -\left(2\sin\frac{\varphi}{2}\right)^6$ defined on $[0, 2\pi]$.

Let us formulate main results. We introduce some functions defined on the interval $\varphi \in (0, \pi)$. Put

$$\beta(\varphi) := \arccos\left(1 - (1 - \cos\varphi)e^{\frac{2\pi i}{3}}\right) \tag{1}$$

(the *arccos* function is multivalued, so $\beta(\varphi)$ is one of its regular branches),

$$\begin{gathered} c(\varphi) := \Re(\beta(\varphi)), \quad b(\varphi) := \Im(\beta(\varphi)), \\ B(\varphi) := \Re(\sin(\beta)e^{\frac{-\pi i}{3}}), \quad C(\varphi) := -\Im(\sin(\beta)e^{\frac{-\pi i}{3}}). \end{gathered} \tag{2}$$

Theorem 1. *Let $\lambda = g(\varphi)$. Then the equation $\det T_n(a - g(\varphi)) = 0$ is equivalent to the following equations:*

$$\tan\left(\frac{n+3}{2}\varphi\right) = f(\varphi, n) \tag{3}$$

and

$$\tan\left(\frac{n+3}{2}\varphi\right) = \frac{1}{h(\varphi, n)}, \tag{4}$$

where

$$f(\varphi, n) = 2\frac{B(\varphi)\sin((n+3)c(\varphi)) + C(\varphi)\sinh((n+3)b(\varphi))}{\sin(\varphi)(\cos((n+3)c(\varphi)) + \cosh((n+3)b(\varphi)))},$$

$$h(\varphi, n) = 2\frac{B(\varphi)\sin((n+3)c(\varphi)) - C(\varphi)\sinh((n+3)b(\varphi))}{\sin(\varphi)(-\cos((n+3)c(\varphi)) + \cosh((n+3)b(\varphi)))},$$

$\varphi \in (0, \pi)$.

Theorem 2. *If n is sufficiently large, then*

1. *Equation* (3) *has exactly one root φ_{2j-1} in each of the intervals $(\frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3})$, where $j \in \{1, \dots, [\frac{n+1}{2}]\}$. Moreover, we can write the following estimate*:

$$\left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}\right| \leqslant \frac{5\pi}{n+3}(0.8)^k, \tag{5}$$

where k is the iteration number.

2. *Equation* (4) *has exactly one root φ_{2j} in each of the intervals $(\frac{2\pi j}{n+3}, \frac{2\pi(j+1)}{n+3})$, where $j \in \{1, \dots, [\frac{n}{2}]\}$. Moreover,we can write the following estimate*:

$$\left|\varphi_{2j}^{(k)} - \varphi_{2j}\right| \leqslant \frac{5\pi}{n+3}(0.8)^k, \tag{6}$$

where k is the iteration number.

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Bifurcations in near-Hamiltonian systems with damped oscillatory perturbations

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Consider the system of two differential equations

$$\frac{dx}{dt} = \partial_y H(x,y) + F(x,y,S(t),t), \quad \frac{dy}{dt} = -\partial_x H(x,y) + G(x,y,S(t),t),$$

where $H(x,y)$, $F(x,y,S,t)$, and $G(x,y,S,t)$ are smooth functions defined for all $\mathbf{z} = (x,y) \in \mathbb{R}^2$, $S \in \mathbb{R}$, $t > 0$, 2π-periodic with respect to S. It is assumed that $H(x,y) = |\mathbf{z}|^2/2 + \mathcal{O}(|\mathbf{z}|^3)$ as $\mathbf{z} \to 0$, $F(x,y,S,t) \to 0$ and $G(x,y,S,t) \to 0$ as $t \to \infty$ for any fixed values of (x,y,S), and $S'(t) \to \text{const}$ as $t \to \infty$. The effect of damped perturbations $F(x,y,S(t),t)$ and $G(x,y,S(t),t)$ on a global behaviour of solutions in the vicinity of the equilibrium of the limiting system is investigated. In particular, possible long-term asymptotic regimes for solutions are described. It is shown that depending on the parameters and structure of perturbations there can be a phase-locking mode with a phase difference tending to a constant at infinity, and a phase-drifting mode with an unboundedly growing phase difference. In both regimes, the equilibrium can remain stable, become asymptotically stable or unstable in the perturbed system. In the case of loss of stability, the trajectories of the perturbed system, starting near the equilibrium, can have an unboundedly growing amplitude at infinity.

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Homogenization of nonstationary Schrödinger-type equations

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Let $\Gamma \subset \mathbb{R}^d$ be a lattice and let Ω be the cell of Γ. Next, $\widetilde{\Gamma}$ is the dual lattice, and $\widetilde{\Omega}$ is the central Brillouin zone of the dual lattice. For Γ-periodic functions, we put $f^\varepsilon(\mathbf{x}) := f(\mathbf{x}/\varepsilon)$, $\varepsilon > 0$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider a selfadjoint matrix strongly elliptic second-order differential operator A_ε given by $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$. Here $g(\mathbf{x})$ is a bounded and positive definite Γ-periodic $(m \times m)$-matrix-valued function. Next, $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$, where b_j are constant $(m \times n)$-matrices. It is assumed that $m \geqslant n$ and the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ has rank n for $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$.

Introduce the effective operator $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$, where g^0 is the so-called effective matrix. Recall the definition of g^0. Suppose that a matrix-valued function $\Lambda(\mathbf{x})$ is a Γ-periodic solution of the problem $b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}) = 0$, $\int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0$. Denote $\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1})$. Then $g^0 = \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}$.

We study the behavior of the operator exponential $e^{-iA_\varepsilon \tau}$ for small ε and $\tau \in \mathbb{R}$. According to [1], as $\varepsilon \to 0$, $e^{-iA_\varepsilon \tau}$ converges to $e^{-iA^0 \tau}$ in the norm of operators acting from the Sobolev space $H^3(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$. The following estimate holds:

$$\left\| e^{-iA_\varepsilon \tau} - e^{-iA^0 \tau} \right\|_{H^3(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(1 + |\tau|)\varepsilon. \tag{1}$$

Our goal is to refine this estimate by taking correctors into account. We managed to approximate not $e^{-iA_\varepsilon \tau}$, but the operator $e^{-iA_\varepsilon \tau}(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)$. Here Π_ε is the pseudodifferential operator whose symbol is the characteristic function of the set $\widetilde{\Omega}/\varepsilon$.

Theorem 1. *We have*

$$\left\| e^{-iA_\varepsilon \tau}(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-iA^0 \tau} - \varepsilon K(\varepsilon, \tau) \right\|_{H^6(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(1 + |\tau|)^2 \varepsilon^2, \tag{2}$$

$$\left\| e^{-iA_\varepsilon \tau}(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-iA^0 \tau} - \varepsilon K_1(\varepsilon, \tau) \right\|_{H^4(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant C(1 + |\tau|)\varepsilon. \tag{3}$$

The correctors $K_1(\varepsilon, \tau)$ and $K(\varepsilon, \tau)$ are given by

$$K_1(\varepsilon, \tau) = \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) e^{-iA^0 \tau}, \quad K(\varepsilon, \tau) = K_1(\varepsilon, \tau) - i \int_0^\tau e^{-iA^0(\tau - \rho)} M(\mathbf{D}) e^{-iA^0 \rho} d\rho.$$

Here $M(\mathbf{D}) := b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})$ and $L(\mathbf{D})$ is the differential operator with the symbol

$$L(\boldsymbol{\xi}) := |\Omega|^{-1} \int_\Omega \left(\Lambda(\mathbf{x})^* b(\boldsymbol{\xi})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\xi}) \Lambda(\mathbf{x}) \right) d\mathbf{x}.$$

Estimates (1)–(3) are order sharp. In the general case, they are sharp with respect to the type of the operator norm and with respect to dependence on τ. However, under some additional assumptions these results are improved.

The results are applied to investigation of the solutions to the Cauchy problem for the equation $i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},\tau) = (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x},\tau)$ with the initial data from a special class.

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Singlarities of solutions to soliton equations represented by L, A, B-triples and the zero level discrete spectra of L-operators

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We discuss the formation of singularities of solutions to $2+1$-dimensional soliton equations. We consider the situation where such a formation appears at the moment when the multiplicity of the zero level discrete spectrum descreases. Such examples were constructed for the Novikov–Veselov, modified Novikov–Veselov and Davey–Stewartson II equations. We consider in detail the Davey-Stewartson II equation. In order to construct explicit examples of such formations, we use the Moutard transformation for solutions of this equation. This transformation is geometrically interpreted by means of the spinor (Weierstrass) representation of surfaces in the four-dimensional Euclidean space. Using the Moutard transformation and minimal surfaces, we construct examples of solutions that have smooth rapidly decreasing initial data and lose their regularity in a finite time [1].

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Integral characteristic equation method to solve a nonlinear eigenvalue problem

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Let $\bar{\mathrm{I}} = [0, h]$, where $h > 0$, let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = (0, +\infty)$ and let $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$ be parameters. In addition, let $a(x) \in C^1(\bar{\mathrm{I}})$ be a given nonnegative function that is not identically zero and satisfies the condition $a'(x) \geqslant 0$.

Problem $\mathcal{P}$ consists in finding eigenvalues $\lambda = \widehat{\lambda}$ such that there exist solutions $u \equiv \widehat{u}(x; \widehat{\lambda}; \alpha) \in C^2(\bar{\mathrm{I}})$ of the equation

$$u'' = -(a(x) - \lambda)u - \alpha u^3, \tag{1}$$

satisfying the boundary conditions

$$u|_{x=0} = 0, \quad u'|_{x=0} = A \neq 0,$$
$$u|_{x=h} = 0,$$

where $(x, \lambda, \alpha) \in \bar{\mathrm{I}} \times \mathbb{R} \times \mathbb{R}_+$ and $A > 0$ is a real constant.

Problem $\mathcal{P}$ describes electromagnetic TE-wave propagation in a plane shielded waveguide filled with inhomogeneous nonlinear medium. From physical point of view, eigenvalues $\widehat{\lambda}$ are propagation constants of the waveguide.

If we set $\alpha = 0$ in (1), then problem $\mathcal{P}$ degenerates into the classical linear Sturm–Liouville problem which has only a finite number of positive and an infinite number of negative solutions [1].

In the present paper, a novel approach called *integral characteristic equation method* is developed and applied to solve problem $\mathcal{P}$. In particular, we prove the following result [2].

Theorem 1. *For any $\alpha > 0$ problem $\mathcal{P}$ has infinitely many nonperturbative positive eigenvalues $\widehat{\lambda}_i$ with accumulation point at infinity.*

Theorem 1 shows that the perturbation approach based on the use of the linearized problem does not allow one to study problem $\mathcal{P}$ even if $\alpha > 0$ is small.

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Solution of the two-dimensional massless Dirac equation with linear potential and localized right-hand side

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We consider the two-dimensional massless Dirac equation

$$x_1\sigma_0\psi + \sigma_1(-ih\psi_{x_1}) + \sigma_2(-ih\psi_{x_2}) = \psi^0\left(\frac{x-x^0}{h}\right),$$

where $x^0 = (-a, 0)$, $a > 0$, $\psi^0(x)$ is a smooth fast-decaying function, $h \ll 1$, σ_j are the Pauli matrices. The solution must satisfy the absorption limit principle. The talk will be devoted to the construction of an asymptotic solution as $h \to 0$. Using the method of [1], we can construct an asymptotic solution outside a neighborhood of the singular line $x_2 = 0$. Earlier in paper [2], the asymptotics of the fundamental solution for the singular ray $x_2 = 0, x_1 > 0$ was obtained.

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Blow-up and global solvability of the Cauchy problem for the equation of nonlinear long longitudinal waves in a viscoelastic rod

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Consider the nonlinear differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 \sigma(u)}{\partial x^2} + \beta^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \alpha \frac{\partial^3 u}{\partial x^2 \partial t}$$

of the Sobolev type, modeling longitudinal waves in an infinite viscoelastic rod and taking into account dispersion and dissipation. Here $u = \partial w/\partial x$ is the longitudinal deformation (w is the longitudinal displacement of the points of the rod along its axis coinciding with the x axis, the term with the coefficient β^2 represents the main

manifestation of dispersion, while the term with the coefficient α describes dissipation), α, β are positive parameters, and $\sigma(u)$ is the total stress of the rod over the cross section. The solvability of the Cauchy problem for this equation in the space of continuous functions on the entire real axis is studied.

An explicit form of the solution of the corresponding linear equation is found.

The time period for the existence and uniqueness of a classical solution of the Cauchy problem for a nonlinear equation is established, and an estimate for the norm of this local solution is obtained.

Conditions for the existence of a global solution and the blow-up of a solution on a finite interval are considered.

On some questions in the theory of elliptic boundary-value problems

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Starting point for the study is the model pseudo-differential equation [1–3]

$$(Au)(x) = v(x), \quad x \in C \tag{1}$$

in the cone $C \subset \mathbb{R}^m$, where $A : H^s(C) \to H^{s-\alpha}(C)$ is a pseudo-differential operator with the symbol $A(\xi)$ satisfying the condition

$$c_1(1+|\xi|)^\alpha \leqslant |A(\xi)| \leqslant c_2(1+|\xi|)^\alpha.$$

Such cone C has certain parameters as a rule, for example $C_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\}$ with parameter a and $C_+^{a,b} = \{x \in \mathbb{R}^3 : x = (x_1, x_2, x_3), x_3 > a|x_1| + |x_2|, a, b > 0\}$ with two parameters a, b. A very natural question arises: if we have a solution of equation (1), then what is its limit value as the parameters tend to their endpoint values 0 or ∞? Some cases were discussed in [4].

One can consider a discrete version of equation (1) using the following constructions for functions of a discrete variable $u_d(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m, h > 0$. Let $C_d = h\mathbb{Z}^m \cap C, \hbar = h^{-1}, \mathbb{T} = [-\pi, \pi]$ and $\tilde{A}_d(\xi)$ be a measurable periodic function in $\mathbb{R}^m$ with basic square of periods $\hbar\mathbb{T}^m$. A digital pseudo-differential operator A_d with the symbol $\tilde{A}_d(\xi)$ in the discrete cone C_d is an operator of the type

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} \tilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in C_d,$$

where $\tilde{u}_d(\xi)$ denotes the discrete Fourier transform of u_d, see [5].

We can introduce discrete analogues of spaces $H^s(C_d)$, and for the special case $C = \mathbb{R}_+^m$, it is possible to obtain solvability conditions for the discrete analogue of equation (1). It was shown that discrete solutions have approximation properties for small h. The similar results were obtained for a discrete quadrant on the plane.

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Stable numerical schemes for modelling incompressible fluid flows in time-dependent domains

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We present a unified numerical approach to finite-element modelling of incompressible fluid flows in time-dependent domains. The approach features relatively large (independent of mesh size) time steps, solution of one linear system per time step, and relatively coarse computational meshes in space. The approach is monolithic and allows standard P_2-P_1 (Taylor–Hood) finite element spaces. It is applicable to the Navier–Stokes equations in time-dependent domains, the fluid-structure interaction (FSI) problems, and the fluid-porous structure interaction (FPSI) problems. The properties of the schemes are shown on several benchmarks and hemodynamic applications.

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Boundary singular problems for mixed quasilinear equations

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We study the existence of solutions to the problem

$$\begin{aligned} -\Delta u + u^p - M|\nabla u|^q &= 0 \qquad \text{in } \Omega \\ u &= \mu \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

in a bounded domain Ω, where $p > 1$, $1 < q < 2$, $M > 0$, μ is a nonnegative Radon measure in $\partial\Omega$, and the associated problem with a boundary isolated singularity at $a \in \partial\Omega$,

$$\begin{aligned} -\Delta u + u^p - M|\nabla u|^q &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \setminus \{a\}. \end{aligned} \tag{2}$$

The difficulty lies in the opposition between the two nonlinear terms which are not on the same nature. Existence of solutions to (1) is obtained under a capacitary condition

$$\mu(K) \leqslant c \min\left\{ cap^{\partial\Omega}_{\frac{2}{p},p'}, cap^{\partial\Omega}_{\frac{2-q}{q},q'} \right\} \quad \text{for all compact sets } K \subset \partial\Omega. \tag{3}$$

Problem (2) depends on several critical exponents on p and q as well as the position of q with respect to $\frac{2p}{p+1}$.

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Spectral analysis of Volterra integro-differential equations and associated semigroups of operators

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We consider abstract linear second-order integro-differential equations in a Hilbert space set on the positive half-line and having unbounded operator coefficients and integral terms of the Volterra convolution type with kernels represented by the Stieltjes integral of a decaying exponential. The equations under study represent an abstract form of partial integro-differential equations arising in the theory of viscoelasticity and have many other important applications. Sufficient conditions are found under which the initial value problem for the equation is well-posed in weighted Sobolev spaces. Localization and structure of the spectra of operator-functions being the symbols of these equations are established. We provide a method for reducing the initial value problem for an equation of this class to the Cauchy problem for a linear differential equation in an extended function space. The existence of a contractive C_0-semigroup is proved. As a corollary, we establish the well-posedness of the resulting Cauchy problem for the first-order differential equation in an extended function space and the initial value problem for the original abstract integro-differential equation, and indicate the relation between their solutions. Then we study the properties of the generator of a semigroup associated with Volterra integro-differential equations in Hilbert spaces. As an example, the results are applied to the case of exponential and fractional exponential kernels (the Rabotnov functions) of integral operators (see [1, 2]).

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Mathematical modelling of respiratory viral infections

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In this lecture we will present an overview of recent works on mathematical modelling of respiratory viral infections at the individual and population levels. We will begin with the investigation of infection progression in cell cultures and in tissues of human body. We will determine viral load and infection spreading speed and we will apply these results to evaluate infectivity and severity of symptoms for different variants of the SARS-CoV-2 infection. Next, using the estimates of the infection transmission rate, we will present new immuno-epidemiological models and will use them to evaluate the epidemiological situation for the ongoing COVID-19 pandemic.

This cycle of works is done in collaboration with L. Ait Mahiout, M. Banerjee, N. Bessonov, S. Ghosh, B. Kazmierczak, A. Mozokhina, A. Tokarev.

The work is supported by the RUDN University Strategic Academic Leadership Program.

Maxwell's equations and Yang–Mills equations in complex variables

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This expository talk first provides a view of Maxwell's equations from the perspective of complex differential forms and the Hodge star operator. The electric field and the magnetic field are complex 3-dimensional in this case. We will see that holomorphic functions naturally give rise to nontrivial solutions to Maxwell's equations. The discussion extends to Yang–Mills (YM) equations, where we will take another look at YM-Lagrangian, YM functional as well as the Belavin-Polyakov-Tyupkin-Schwartz instanton solution.

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Smooth solutions of hyperbolic differential-difference equations in a half-space

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In the present paper, we study the existence of smooth solutions of two hyperbolic differential-difference equations in the half-space $\{(x,t)|\, x \in \mathbb{R}^n,\ t > 0\}$.

The first of these equations contains superpositions of differential operators and shift operators with respect to each of the spatial variables,

$$u_{tt}(x,t) = a^2 \sum_{j=1}^{n} u_{x_j x_j}(x,t) + \sum_{j=1}^{n} b_j u_{x_j x_j}(x_1, \dots, x_{j-1}, x_j - h_j, x_{j+1}, \dots, x_n, t),$$

where $a > 0$, $b_1, \dots, b_n$, and $h_1, \dots, h_n$ are given real numbers.

The second equation contains a sum of differential operators and shift operators with respect to each of the spatial variables,

$$u_{tt}(x,t) = c^2 \sum_{j=1}^{n} u_{x_j x_j}(x,t) - \sum_{j=1}^{n} d_j u(x_1, \dots, x_{j-1}, x_j - l_j, x_{j+1}, \dots, x_n, t),$$

where $c > 0$, $d_1, \dots, d_n$, and $l_1, \dots, l_n$ are given real numbers.

Three-parameter families of solutions are constructed for these equations. In this case, some ideas of papers [1–3] were used.

We prove theorems showing that the solutions obtained are classical ones provided that the real parts of the symbols of the corresponding differential–difference operators are positive. Classes of equations for which these conditions are satisfied are given. Detailed results of the study are published in [4].

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Peculiarity of solutions of Laplace equation as applied to the problem of describing the motion of a hydrodynamic discontinuity in a potential and incompressible flow in an external region

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When considering the motion of hydrodynamic discontinuities, a potential and incompressible flow in front of the surface of discontinuityis is usually assumed. This creates a mathematical feature that can fundamentally change the idea of the motion of discontinuities propagating in potential and incompressible flows. It is well known that for vanishingly low viscosity, the integral relation on the boundary (Dirichlet, Neumann problems) connects the tangential and normal components of the velocity. Knowing one of them along the boundary of the discontinuity, one can determine the entire external flow. However, assuming the external flow is smooth, this will also be the case for all derivatives of velocity with respect to coordinates and time. Then a paradox arises, knowing the position of the discontinuity and the velocity data at a point on its surface, it is possible to determine the motion of this discontinuity without taking into account the influence of the flow behind the front, as well as the entire external flow. There is no physical explanation for this mechanism. It is possible that a boundary layer is formed in front of the front, where viscosity plays a significant role and the Euler equations are violated.

As is known, a harmonic function is infinitely differentiable within the domain of definition. Let us single out a certain region in the external flow so that it lies entirely outside the front. Then, differentiating the potential of the fluid velocity by coordinates and time, one can get on its boundary an infinite number of integro-differential relations for only two surface unknowns $\partial\varphi/\partial n$ and φ, where φ is the fluid velocity potential. It is possible that not all of these equations are independent. Some are complex consequences of the others.

A new symmetric interior penalty discontinuous Galerkin formulation for the Serre–Green–Naghdi equations

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In this oral presentation we will talk about the construction of a new discontinuous Galerkin discrete formulation to approximate the solution of the Serre–Green–Naghdi (SGN) equations in the one-dimensional horizontal framework. Such equations describe the time evolution of shallow water free surface flows in the fully nonlinear

and weakly dispersive asymptotic approximation regime. A new non conforming discrete formulation belonging to the family of symmetric interior penalty discontinuous Galerkin methods (SIP-DG) is introduced to accurately approximate the solutions of the second-order elliptic operator occurring in the SGN equations [2]. We show that the corresponding discrete bilinear form enjoys some consistency and coercivity properties, thus ensuring that the corresponding discrete problem is well-posed. The resulting global discrete formulation is then validated through an extended set of benchmarks, including convergence studies and comparisons with data taken from experiments. This formulation can be extended to the SGN equations with vorticity in the one-dimensional horizontal case [1]. Such equations have the interesting feature of accounting for the full vertical dynamics of the vorticity, while being defined only in terms of flow variables that do not have any vertical dependency. We perform several numerical validations, concerning the propagation, transformations, and interactions of stable rotational solitary waves.

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On the index of differential-difference operators in an infinite cylinder

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In the infinite cylinder $M = \mathbb{S}^1 \times \mathbb{R}$ with coordinates (x, t), the action of the group $\mathbb{Z}$ is given by the diffeomorphisms $g^k \colon M \to M$, $k \in \mathbb{Z}$, where $g(x, t) = (x, t + 2\pi)$. We consider the following operator in M:

$$D = \sum_k D_k T^k \colon H^{s,\gamma^-,\gamma^+}(M, \mathbb{C}^N) \longrightarrow H^{s-m,\gamma^-,\gamma^+}(M, \mathbb{C}^N), \tag{1}$$

where D_k is a matrix differential operator of order $\leqslant m$ in M, $T^k u(x, t) = u(x, t-2\pi k)$ is a shift operator with respect to t, and $H^{s,\gamma^-,\gamma^+}(M)$ is a weighted Sobolev space (see [1]). Here we assume that only a finite number of terms in sum (1) are nonzero, and the coefficients D_k do not depend on t for large t.

The *interior symbol* of operator (1) at a point $(x, t, \xi, p) \in T_0^* M = \{(x, t, \xi, p) \mid \xi^2 + p^2 \neq 0\}$ of the cotangent bundle without the zero section is an operator-valued function $\sigma \colon \ell^2(\mathbb{Z}, \mu) \otimes \mathbb{C}^N \to \ell^2(\mathbb{Z}, \mu) \otimes \mathbb{C}^N$ given by the formula

$$\sigma(D)(x, t, \xi, p) = \sum_k \sigma(D_k)(x, t + 2\pi n, \xi, p)\mathcal{T}^k, \tag{2}$$

where $\sigma(D_k)$ is the principal symbol of D_k and $\mathcal{T} w(n) = w(n-1)$ is the shift operator.

The *conormal symbol* of operator (1) is a pair (σ_c^+, σ_c^-) of the parameter-dependent families

$$\sigma_c^{\pm}(D)(p) = \sum_k D_k^{\pm}(p) e^{ikp}, \text{ where } D_k^{\pm}(p) = D_k^{\pm}(x, -i\partial/\partial x, \pm\infty, p) \tag{3}$$

are the conormal symbols of $D_k(x, -i\partial/\partial x, t, -i\partial/\partial t)$ at $t = \pm\infty$, respectively.

Operator (1) is called *elliptic* if operator (2) is invertible for all $(x, t, \xi, p) \in T_0^*M$, and operators (3) are invertible on the weight lines $L_{\gamma^\pm} = \{p \in \mathbb{C} \mid \operatorname{Im} p = \gamma^\pm\}$. Moreover, it is proved in [2] that operator (1) is invertible for large $p \in L_{\gamma^\pm}$ if its internal symbol (2) is elliptic.

We obtained an index formula for elliptic operators (1) in terms of symbols (2) and (3). In particular, we defined the η-invariant [3] of families (3) taking into account the contribution of infinity to the index formula. In the absence of shifts (i.e. $D = D_0$ in (1)), our formula yields the Fedosov–Schulze–Tarkhanov formula (cf. [4]).

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Нелокальная краевая задача Стеклова первого класса для уравнения теплопроводности

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В работе В. А. Стеклова [1, с. 67] для уравнения теплопроводности изучалась нелокальная краевая задача

$$\begin{cases} a_1 u(a,t) + a_2 u_x(a,t) + a_3 u(b,t) + a_4 u_x(b,t) = 0, \\ b_1 u(a,t) + b_2 u_x(a,t) + b_3 u(b,t) + b_4 u_x(b,t) = 0, \end{cases} \tag{1}$$

где a_k, b_k, $k = 1, 2, 3, 4,$ — некоторые заданные числа.

Нелокальные краевые условия (1) распадаются на два случая. В первом случае система (1) разрешима относительно производных, т. е. $a_2 b_4 - a_4 b_2 \neq 0$. Здесь нелокальные условия (1) могут быть представлены в виде

$$u_x(a,t) = \alpha u(a,t) + \beta u(b,t), \quad -u_x(b,t) = \gamma u(a,t) + \delta u(b,t).$$

В работе [1, стр. 67] эти условия были названы условиями первого класса.

Во втором случае система (1) не разрешима относительно производных, т. е. $a_2b_4 - a_4b_2 = 0$. В этом случае нелокальные условия (1) могут быть представлены в виде

$$u(a,t) = \alpha u(b,t), \quad u_x(b,t) = \beta u_x(a,t) + \gamma u(b,t).$$

Эти условия были названы условиями второго класса. Для решения уравнения теплопроводности с нелокальными краевыми условиями Стеклова второго класса для случаев 1) $|\alpha| < 1$, $|\beta| < 1$ и $\gamma \leqslant 0$ или 2) $|\alpha| > 1$, $|\beta| > 1$ и $\alpha\beta\gamma \leqslant 0$, были построены априорные оценки в работе [2].

В работе [1, с. 69] были построены априорные оценки для решения уравнения теплопроводности с нелокальными условиями первого класса в случае

$$4\alpha\delta - (\beta+\gamma)^2 > 0, \quad \alpha > 0.$$

В данной работе получены априорные оценки решения нелокальной краевой задачи Стеклова первого класса для уравнения теплопроводности с более широким охватом параметров α, β, γ и δ, удовлетворяющих следующим условиям:

$$\alpha\delta - \beta\gamma > 0, \quad (\alpha-\delta)^2 - (\beta-\gamma)^2 > 0, \quad \alpha+\delta > 0.$$

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Вырожденное линейное уравнение с несколькими дробными производными Герасимова—Капуто

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Рассмотрим линейное неоднородное уравнение дробного порядка

$$D_t^{\alpha} Lx(t) = \sum_{k=1}^{n} D_t^{\alpha_k} M_k x(t) + g(t), \tag{1}$$

где D_t^{β} — дробная производная Герасимова—Капуто порядка $\beta > 0$ [2], в случае $\ker L \neq \{0\}$, где $L, M_k \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ (линейные ограниченные операторы), $k = 1, 2, \ldots, n-1$, $M_n \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$ (линейный замкнутый оператор), $0 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha$, $m = \lceil\alpha\rceil$, $m_k = \lceil\alpha_k\rceil$, $k = 1, 2, \ldots, n$, $g \in C([0,T];\mathcal{Y})$.

Если M_n $(L,0)$-ограничен, то существуют проекторы

$$P = \frac{1}{2\pi i}\int\limits_{|\mu|=R} (\mu L - M_n)^{-1} L d\mu \in \mathcal{L}(\mathcal{X}), \quad Q = \frac{1}{2\pi i}\int\limits_{|\mu|=R} L(\mu L - M_n)^{-1} d\mu \in \mathcal{L}(\mathcal{Y})$$

при таком $R > 0$, что $\sigma^L(M_n) \subset \{\mu \in \mathbb{C} : |\mu| \leqslant R\}$ [1]. Решением задачи

$$x^{(l)}(0) = x_l,\ l = 0, 1, \ldots, m_n - 1,\ (Px)^{(l)}(0) = x_l,\ l = m_n, m_n + 1, \ldots, m - 1, \quad (2)$$

для уравнения (1) будем называть функцию $x : [0, T] \to \mathcal{X}$, для которой $x \in C^{m_n - 1}([0, T]; \mathcal{X})$, $D_t^\alpha Lx, D_t^{\alpha_k} M_k x \in C([0, T]; \mathcal{Y})$, $k = 1, 2 \ldots, n$, выполняются равенства (1) при всех $t \in [0, T]$ и (2).

Теорема 1 (см. [2]). *Пусть $L, M_k \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $k = 1, 2, \ldots, n - 1$, $M_n \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ $(L, 0)$-ограничен, $M_k P = Q M_k$, $k = 1, 2, \ldots, n - 1$, $g \in C([0, T]; \mathcal{Y})$, $x_l \in \mathcal{X}$ при $l = 0, 1, \ldots, m_n - 1$, $x_l \in \mathcal{X}^1$, $l = m_n, m_n + 1, \ldots, m - 1$. Тогда существует единственное решение задачи* (1), (2).

Замечание 1. Если оператор M_n $(L, 0)$-ограничен, условия $(Px)^{(l)}(0) = x_l \in \mathcal{X}^1$ эквивалентны условиям $(Lx)^{(l)}(0) = y_l = Lx_l \in \mathcal{Y}^1$ при $l = m_n, m_n + 1, \ldots, m - 1$.

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Коллокационно-вариационные подходы к решению интегральных уравнений Вольтерра I рода

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Многие прикладные задачи используют для математической постановки интегральные модели, где ключевую роль играет уравнение Вольтерра I рода [1,2]. В данной работе рассматривается классическое уравнение Вольтерра I рода на конечном отрезке. Предполагается, что ядро и правая часть уравнения удовлетворяют стандартным требованиям. Для создания и обоснования численных методов решения таких уравнений важную роль играет понятие их степени неустойчивости [2], близкое понятию индекса для дифференциально-алгебраических уравнений (ДАУ) [3]. Поэтому некоторые подходы, развитые для численного решения ДАУ [4,5], можно применять и для численного решения интегральных уравнений Вольтерра I рода.

Для решения уравнений Вольтерра I рода со степенью неустойчивости 1 мы предлагаем одношаговые методы, в основу построения которых положены двухшаговые квадратурные методы. Алгоритм численного метода включает этап минимизации нормы приближенного решения в некоторых аналогах пространства Соболева, что обеспечивает устойчивость предложенных методов. Теоретические выкладки подтверждаются результатами расчетов для известных тестовых примеров [6].

Работа выполнена при финансовой поддержке РФФИ (код проекта № 20-51-S52003 МНТ_а).

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Вывод уравнений электродинамики и гравитации из принципа наименьшего действия Гильберта—Эйнштейна—Паули

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В работе описано получение уравнений электродинамики и гравитации в замкнутой форме из принципа наименьшего действия в форме системы уравнений типа Власова [1–3]. Проясняется смысл уравнений типа Власова: это единственный пока способ получить и уравнение гравитации, и уравнения электродинамики из принципа наименьшего действия. А также единственный пока способ замкнуть систему уравнений гравитации и электродинамики с помощью принципа наименьшего действия, используя функцию распределения объектов (электронов, ионов, звёзд в галактиках, галактик в супергалактиках или Вселенной)

по скоростям и пространству. Соответствующие уравнения гидродинамического уровня (например, уравнения магнитной гидродинамики) также естественно получать из уравнений типа Власова гидродинамической подстановкой (пока единственный способ связи с классическим действием и для этих уравнений) [4, 5]. Представляет значительный интерес исследовать различные классы решений полученных уравнений, как это делалось в [6, 7]. Особый интерес должно представлять асимптотическое поведение решений уравнений Власова, и тут могла бы помочь его аналогия с уравнением Лиувилля [8]. Мы показали также, что полученные уравнения типа Власова должны быть применены к объяснению эволюции Вселенной, так как именно из уравнений Власова—Пуассона следуют нерелятивистские аналоги решений Фридмана, решения Милна—Мак-Кри [9, 10]. При этом они являются точным следствием уравнений Власова—Пуассона, поэтому получаются без эвристических предположений работ [9, 10] и обосновывают и обобщают их. Эти решения позволили выяснить роль лямбда-члена, его эквивалентность потенциалу Гурзадяна и эквивалентность этого любой однородной субстанции, связанной с решением уравнения Пуассона. Правая часть уравнения Эйнштейна дает надежду на объяснение ускоренного расширения Вселенной без этих дополнительных предположений.

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Энтропийное решение для уравнения с сингулярным потенциалом в гиперболическом пространстве

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Рассмотрим задачу

$$-\mathrm{div}_g(a(x,u,\nabla_g u)) + \mathcal{B}u = f, \quad u|_{\partial\mathbb{H}^n} = 0, \quad f \in L_1(\mathbb{H}^n). \tag{1}$$

Будем рассматривать оператор вида

$$\mathcal{B}u = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_k}(F_i^k g^{ij} u_{x_j}\sqrt{g}) + |u|^{p_0(x)-2}u + b(x,u,du) = \Lambda u + |u|^{p_0(x)-2}u + b(x,u,du),$$

где функции $F_i^k(x)$ принадлежат $L_{1,\mathrm{loc}}(\mathbb{H}^n)$, и при любом $x \in \mathbb{H}^n$ элементы $F_i^k(x)$ задают неотрицательно определенную матрицу. Действие оператора $\Lambda : \mathcal{D}(\mathbb{H}^n) \to \mathcal{D}'(\mathbb{H}^n)$ определяется формулой

$$\langle \Lambda u, v\rangle = \int\limits_{B_1} (F_i^k g^{ij} u_{x_j}\sqrt{g} v_{x_k})dx = \int\limits_{\mathbb{H}^n} (F\nabla_g u, dv)d\nu\ ,\ u,v \in \mathcal{D}(\mathbb{H}^n). \tag{2}$$

Векторное поле a в уравнении (1) удовлетворяет условиям ограниченности с $q < p(x)$

$$|a(x,u,y)|_g^{p'(x)} \leqslant C(G(x) + |u|^q + |y|^{p(x)}), \quad G(x) \in L_1(\mathbb{H}^n), \tag{3}$$

коэрцитивности

$$(a(x,u,y),y) \geqslant c_0|y|^{p(x)} - G(x), \quad u \in \mathbb{R}, \quad x \in \mathbb{H}^n, \quad y \in T_x^*\mathbb{H}^n, \quad c_0 > 0, \tag{4}$$

монотонности

$$(a(x,u,y) - a(x,u,z), y - z) > 0,\ y \neq z. \tag{5}$$

Кроме того, пусть справедливы неравенства

$$|b(x,u,y)| \leqslant \widehat{C}\left(|y|^{p(x)} + G(x)\right), \quad (b(x,u,y),y) \geqslant 0. \tag{6}$$

Пространство $\mathbb{B}_{p(\cdot)}(\mathbb{H}^n)$ — пополнение $\mathcal{D}(\mathbb{H}^n)$ по норме

$$\|u\|_{p(\cdot),\Lambda} = \|u\|_{p(\cdot),1} + \sqrt{\langle \Lambda u, u\rangle}.$$

Определение 1. *Энтропийным решением* задачи Дирихле называется функция u такая, что $T_k(u) \in \mathbb{B}_{p(\cdot)}(\mathbb{H}^n)$ при любом $k > 0$

1) $B(x) = b(x,u,du) \in L_1(\mathbb{H}^n)$;
2) при всех $k > 0$, $\xi \in C_0^1(\mathbb{H}^n)$ справедливо неравенство

$$\int\limits_{\mathbb{H}^n} (a(x,u,du), d(T_k(u-\xi)))d\nu + \langle \Lambda u + |u|^{p_0(x)-2}u + b(x,u,du) - f, T_k(u-\xi)\rangle \leqslant 0. \tag{7}$$

Теорема 1. *Пусть выполнены условия на a, F, b. Тогда существует энтропийное решение задачи Дирихле.*

Спектральные свойства дифференциально-разностных операторов на конечном интервале

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Теория дифференциальных уравнений с отклоняющимся аргументом ведет свою историю с середины прошлого века. Среди всех таких задач следует выделить задачи, в которых сдвиги присутствуют в старших членах уравнений. Такие задачи рассматривались, например, в работах [1, 2]. Было обнаружено, что, в отличие от обыкновенных дифференциальных уравнений, гладкость решений дифференциально-разностных уравнений может нарушаться внутри интервала, что связано со сдвигами точек границы внутрь интервала.

В дальнейшем подобные задачи изучались в работах [3–6]. Так, например, в работах [5, 6] были получены условия для сохранения гладкости решений уравнения для дифференциально-разностного оператора для произвольной правой части. Мы рассматриваем задачу на собственные значения и собственные функции дифференциально-разностного оператора с постоянными коэффициентами на интервале целой длины с краевыми условиями Дирихле. Отличием от предыдущих работ является тот факт, что вид правой части в такой задаче отнюдь не произволен, что не позволяет применять методы, использующиеся в предыдущих работах. Нами получены достаточные условия существования решений (собственных функций), гладкость которых нарушается внутри интервала.

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Моделирование фазовых переходов в подвижных средах

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В докладе представлено исследование фазового перехода в задаче обтекания твердой фазы вещества его жидкой фазой (например, лед—вода) при достаточно больших числах Рейнольдса. Эта задача разделяется условно на две: задача обтекания поверхности и задача о фазовом переходе.

Предполагается, что на границе раздела фаз y_s есть малая локализованная неровность, которая в начальный момент времени имеет вид

$$y_s = \varepsilon^{4/3}\mu\big((x - x_0)/\varepsilon\big),$$

где $\varepsilon = \mathrm{Re}^{-1/2}$ — малый параметр, $\mu(z) \in \mathcal{S}(\mathbb{R}^1)$, см. рис. 1. Такой выбор начальной формы неровности приводит к двухпалубной структуре пограничного слоя обтекающего ее потока, состоящей из классического пограничного слоя Прандтля и тонкого погранслоя внутри него, см. области II и I на рис. 1, соответственно.

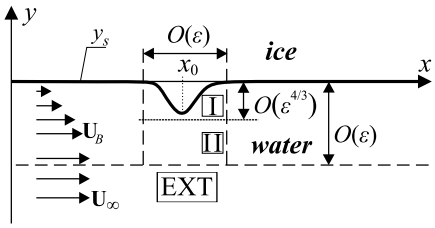

Рис. 1: Геометрия рассматриваемой задачи в начальный момент времени, $\mathbf{U}_B$ — течение Блаизуса, $\mathbf{U}_\infty$ — плоскопараллеьный поток

Фазовый переход рассматривается в рамках модели фазового поля. Это позволяет избежать отдельного вычисления точного положения границы раздела фаз — она, с точностью до параметра регуляризации в системе фазового поля, описывается линией нулевого уровня функции порядка, определяемой во всей области решения задачи. Это дает возможность использовать традиционные разностные схемы для решения задач, возникающих при асимптотической редукции.

Таким образом, задача решается комбинированием известной техники описания двухпалбуной структуры пограничного слоя и результатов асимптотического исследования системы фазового поля. Решение задачи обтекания с двухпалубной структурой при заданном законе изменении во времени формы неровности на пластине изложено в [1], в том числе приведены результаты численного моделирования течения для различных типов этих законов.

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Интегро-дифференциальные уравнения с ограниченными операторами в банаховых пространствах и их приложения

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Пусть $\mathcal{X}$ — банахово пространство, $A \in \mathcal{L}(\mathcal{X})$, т. е. A — линейный ограниченный оператор, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$. Определим интегро-дифференциальный оператор типа Герасимова

$$D^{K,m}x(t) := J^K(D^m x)(t) := \int_0^t K(t-s)x^{(m)}(s)ds,$$

где D^m — производная порядка m.

Теорема. *Пусть $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, существует преобразование Лапласа $\widehat{K}$, которое является однозначной аналитической функцией в $\Omega_{R_0} := \{\mu \in \mathbb{C} : |\mu| > R_0, |\arg \mu| < \pi\}$ при некотором $R_0 > 0$ и удовлетворяет условию*

$$\exists \chi > 0 \quad \exists c > 0 \quad \forall \lambda \in \Omega_{R_0} \quad \|\widehat{K}(\lambda)\|_{\mathcal{L}(\mathcal{X})} > c|\lambda|^{\chi-1}, \tag{1}$$

при этом для всех $\lambda \in \Omega_{R_0}$ существует $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$. Тогда для всех $x_0, x_1, \ldots, x_{m-1} \in \mathcal{X}$ существует единственное решение задачи Коши $x^{(k)}(0) = x_k \in \mathcal{X}$, $k = 0, 1, \ldots, m-1$, для уравнения $(D^{K,m}x)(t) = Ax(t) + f(t)$ при $t \in [0, T]$, где $f \in C([0, T]; \mathcal{X})$. Решение имеет вид

$$x(t) = \sum_{k=0}^{m-1} Y_k(t)x_k + \int_0^t X_{m-1}(t-s)f(s)ds,$$

$$Y_k(t) = \frac{1}{2\pi i} \int_{\partial\Omega_R} (\lambda^m \widehat{K}(\lambda) - A)^{-1} \widehat{K}(\lambda)\lambda^{m-1-k} e^{\lambda t} d\lambda, \quad k = 0, 1, \ldots, m-1,$$

$$X_{m-1}(t) = \frac{1}{2\pi i} \int_{\partial\Omega_R} (\lambda^m \widehat{K}(\lambda) - A)^{-1} \lambda^{m-1} e^{\lambda t} d\lambda, \quad R > R_0.$$

Пример 1. При $K(s) = Is^{\alpha-1}/\Gamma(\alpha)$ имеем дробную производную Герасимова — Капуто.

Пример 2. Пусть $a \in \mathbb{R}$, $\alpha > 0$, $\beta \in (0,1)$, $K(s) = s^{-\beta} E_{\alpha,1-\beta}(as^{\alpha})I$, где $E_{\alpha,1-\beta}$ — функция Миттаг-Леффлера. Тогда $\widehat{K}(\lambda) = \lambda^{\alpha+\beta-1}(\lambda^{\alpha} - a)^{-1} I$ удовлетворяет условию (1) с $\chi \in (0,\beta)$ и обратима при $|\lambda| > a^{1/\alpha}$. В этом случае получаем дробную производную Прабхакара [1].

Работа выполнена в рамках проекта по гранту Президента РФ для поддержки ведущих научных школ НШ-2708.2022.1.1.

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О скорости стабилизации решения задачи Коши для параболического уравнения второго порядка с растущими старшими коэффициентами

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В E_+^{N+1}, $N \geqslant 3$, рассмотрим задачу Коши для параболического уравнения 2-го порядка с младшими коэффициентами и начальной функцией $u_0(x)$

$$Lu = 0, \quad (x,t) \in E_+^{N+1}, \quad u(x,0) = u_0(x). \tag{1}$$

Пусть коэффициенты $a_{ik}(x,t)$ уравнения (1) удовлетворяют условию параболичности

$$\lambda_0^2 b(|x|)|\xi|^2 \leqslant \sum_{i,k=1}^{N} a_{ik}(x,t)\xi_i\xi_k \leqslant \lambda_1^2 b(|x|)|\xi|^2, \tag{2}$$

где

$$b(|x|) = \max(1, |x|^2), \tag{3}$$

$$\sum_{i=1}^{N} |b_i(x,t)| \leqslant B \max(1, |x|), \tag{4}$$

и выполнено условие (C):

$$c(x,t) \leqslant -\beta^2 \tag{5}$$

для любого (x,t) в E_+^{N+1}.

Начальная функция

$$u_0(x) \in C(E^N), |u_0(x)| \leqslant M(1 + |x|^m), \quad m > 0, \quad x \in E^N. \tag{6}$$

Теорема 1.

1. *Если* $u_0(x) \in C(E^N)$ *удовлетворяет (6),* $C(x,t)$ *удовлетворяет условию* (C) *при* $\beta^2 > \lambda_1^2 m(m+S-2) = \beta_0^2$, где

$$S = \frac{\lambda_1^2(N-1) + \lambda_0^2 + B}{\lambda_0^2},$$

то решение задачи (1) имеет предел

$$\lim_{t \to +\infty} u(x,t) = 0 \tag{7}$$

равномерно относительно x *на каждом компакте* K *в* E^N.

2. *Если*

$$\beta^2 > \lambda_1^2(S-1),$$

то для любой $u_0(x) \in C(E^N)$ *и ограниченной в* E^N *справедлива оценка для решения*

$$|u(x,t)| \leqslant M t^{-\frac{\Delta}{3}}, \quad M > 0, \quad t \geqslant t_1 > 0, \tag{8}$$

равномерно по x *на каждом компакте* K *в* E^N,

$$\Delta = \frac{2 - S + \sqrt{D_1}}{2}, \quad D_1 = (2-S)^2 + 4\beta^2, \quad \overline{\beta} = \frac{\beta}{\lambda_1}.$$

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О некоторых нелокальных задачах теории поля на плоскости

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Предлагается рассмотреть ряд нелокальных краевых задач теории поля на плоскости, определённых ядрами функционалов следа в пространстве Соболева $W_2^1(G)$.

В случае регулярного функционала следа примером такой задачи является задача

$$-\Delta \mathbf{u}(\mathbf{x}) = \mathbf{h}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2,$$

$$\int_\Gamma (\mathbf{u}, \mathbf{n}) d\gamma = 0,$$

$$\left.\frac{\partial \mathbf{u}}{\partial n}\right|_\Gamma = \alpha \mathbf{n}(\gamma), \gamma \in \Gamma,$$

где $\mathbf{n}(\gamma)$ — вектор нормали на границе области Γ. В этой задаче искомыми неизвестными являются вектор-функция $\mathbf{u}(\mathbf{x}) \in W_2^1(G)$ и число α.

В случае сингулярного следа для постановки задачи существенна теорема о следе, являющаяся «плоским» вариантом существования следа линейной комбинации

$$\mathbf{D}_\Gamma \mathbf{u} \equiv \frac{\partial \mathbf{u}}{\partial n} - [\mathrm{rot}\mathbf{u}; \mathbf{n}] - (\mathrm{div}\mathbf{u})\,\mathbf{n}$$

в случае трехмерных полей.

Для плоских полей $\mathbf{u} = (u_1; u_2)$ такой сингулярный след имеет линейная комбинация

$$\mathbf{D}_\Gamma \mathbf{u} \equiv \frac{\partial \mathbf{u}}{\partial n} - (\mathrm{rot}_{\mathrm{pl}}\mathbf{u})\,\boldsymbol{\tau} - (\mathrm{div}\mathbf{u})\,\mathbf{n},$$

где $\mathrm{rot}_{\mathrm{pl}}\mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ — плоский оператор Грина, а $\vec{\tau} = (\tau_1; \tau_2)$ — касательный вектор, согласованный с нормалью $\mathbf{n}$ формулами $\tau_1 = -n_2, \tau_2 = n_1$.

Характерными примерами краевых задач, отвечающих ядру оператора $\mathbf{D}_\Gamma$, являются следующие задачи:

1.

$$-\Delta \mathbf{u}(\mathbf{x}) = \mathbf{h}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2,$$
$$\int\limits_\Gamma \left[\left(\frac{\partial \mathbf{u}}{\partial n}, \boldsymbol{\tau}\right) - \mathrm{rot}_{\mathrm{pl}}\mathbf{u}\right] d\gamma = 0,$$
$$\left.\frac{\partial \mathbf{u}}{\partial n}\right|_\Gamma = \alpha \boldsymbol{\tau}(\gamma), \gamma \in \Gamma,$$

где α — также искомое число.

2.

$$-\Delta \mathbf{u}(\mathbf{x}) = \mathbf{h}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2,$$
$$\int\limits_\Gamma \left[\left(\frac{\partial \mathbf{u}}{\partial n}, \mathbf{n}\right) - \mathrm{div}\mathbf{u}\right] d\gamma = 0,$$
$$\left.\frac{\partial \mathbf{u}}{\partial n}\right|_\Gamma = \alpha \mathbf{n}(\gamma), \gamma \in \Gamma.$$

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Стохастическая модель боевых действий

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Рассматривается математическая модель (Ланчестера) боевых действий двух противоборствующих сторон

$$\frac{dx}{dt} = \varepsilon_1(t)x, \tag{1}$$

$$\frac{dy}{dt} = -\varepsilon_2(t)x + \varepsilon_3(t)y, \tag{2}$$

$$x(0) = x_0, y(0) = y_0. \tag{3}$$

Здесь t — время, $x(t), y(t)$ — численности сторон в момент времени t, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ — случайные процессы и x_0, y_0 — случайные величины, не зависящие от $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Поскольку система (1), (2) зависит от случайных процессов $\varepsilon_1, \varepsilon_2, \varepsilon_3$, решение также является векторным случайным процессом. Наибольший интерес представляют математическое ожидание $E[x(t)], E[y(t)]$ и дисперсионные функции $D[x(t)], D[y(t)]$ решения.

Предполагается, что случайные процессы $\varepsilon_1, \varepsilon_2, \varepsilon_3$ заданы характеристическим функционалом, который определяется равенством

$$\psi(u_1, u_2, u_3) = E[\exp(i\sum_{j=1}^{3}\int_0^{t_1} \varepsilon_j(s)u_j(s)ds)],$$

где $[0, t_1]$ — отрезок времени, на котором изучается задача, E обозначает взятие среднего значения по функции распределения процессов $\varepsilon_1, \varepsilon_2, \varepsilon_3$; u_j — суммируемые на $[t_0, t_1]$ функции. Пусть $\chi(t_0, t, s)$ — функция переменной $s \in \mathbb{R}$, определенная по правилу: $\chi(t_0, t, s) = \mathrm{sign}(s - t_0)$, если s принадлежит отрезку $[\min\{t_0, t\}, \max\{t_0, t\}]$, и равна нулю в противном случае.

Теорема 1. *Если функционал ψ имеет первые и вторые вариационные производные, то математическое ожидание решения задачи* (1)–(3) *имеет вид*

$$E[x(t)] = E[x_0]\psi(-i\chi(0, t), 0, 0),$$

$$E[y(t)] = E[y_0]\psi(0, 0, -i\chi(0, t)) + iE[x_0]\int_0^t \frac{\delta}{\delta u_2(s)}\psi(-i\chi(0, s), 0, -i\chi(s, t))ds.$$

Оценка воздействия неконтролируемых факторов на (квази)динамическую систему

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Рассматривается задача описания фазового состояния (квази)динамической системы при воздействии неконтролируемых факторов. Предполагаем гладкость

и ограниченность таких факторов. Подобные задачи возникают в динамике ЛА, прогнозировании параметров погоды, в моделировании биржевых показателей в задачах экономики и др.

В зависимости от математического описания системы, применяются различные методы оценивания, разработанные автором. Для систем, описываемых ОДУ (в зависимости от постановки задачи), строятся гарантирующие, вероятностные методы и соответствующие алгоритмы оценивания влияния неконтролируемых факторов на интегральную воронку системы.

Для систем, описываемых нейросетями, строится ситуационный прогноз с помощью моделирования неконтролируемых факторов.

Если в некоторые моменты времени известно фазовое состояние системы, то описанные оценки улучшаются.

Рассматриваются примеры из динамики ЛА (оценивание ветрового воздействия) и нейросетевого прогнозирования динамики российских индексов акций с учетом различных ситуаций в геополитике.

К задаче о нормальных колебаниях смеси двух жидкостей

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Рассмотрим ограниченную область $\Omega \subset \mathbb{R}^3$ с бесконечно гладкой границей $\partial\Omega$, полностью заполненную гомогенной смесью двух сжимаемых жидкостей. Введём систему координат $Ox_1x_2x_3$, жёстко связанную с областью Ω, таким образом, что ось Ox_3 направлена против действия силы тяжести $-g\mathbf{e}_3$, $g > 0$, а начало координат находится в области Ω. Подставляя в уравнения свободных колебаний рассматриваемой системы решения, зависящие от времени, как $\exp(-\lambda t)$, придём к следующей задаче на собственные значения:

$$\frac{1}{\rho_{l0}}\sum_{k=1}^{2}\left(\mu_{lk}\Delta\mathbf{u}_k + (\mu_{lk}+\lambda_{lk})\nabla\mathrm{div}\mathbf{u}_k\right) + \frac{a}{\rho_{l0}}\sum_{k=1}^{2}(\mathbf{u}_k - \mathbf{u}_l) - \nabla\left(\frac{c_l^{1/2}\rho_l}{\rho_{l0}^{1/2}}\right) = -\lambda\mathbf{u}_l,$$
$$-\frac{c_l^{1/2}}{\rho_{l0}^{1/2}}\,\mathrm{div}(\rho_{l0}\mathbf{u}_l) = -\lambda\rho_l \quad (x\in\Omega), \qquad \mathbf{u}_l = \mathbf{0} \quad (x\in\partial\Omega), \qquad l = 1,2. \tag{1}$$

Здесь $\mathbf{u}_l(x)$ $(x = (x_1, x_2, x_3) \in \Omega)$ — поле скоростей жидкости смеси, $c_l > 0$ и $a > 0$ — фиксированные константы, $\rho_{l0}(x_3) = \rho_{l0}(0)\exp(-gc_l^{-1}x_3)$ — плотность жидкости смеси в состоянии равновесия, $c_l^{-1/2}\rho_{l0}^{1/2}\rho_l(x)$ — динамическая плотность жидкости смеси. Матрицы вязкостей $\{\mu_{jk}\}_{j,k=1}^2$, $\{\lambda_{jk}\}_{j,k=1}^2$ удовлетворяют условиям

$$\{\mu_{jk}\}_{j,k=1}^2 > 0, \quad \{2\mu_{jk}+\lambda_{jk}\}_{j,k=1}^2 > 0.$$

Теорема 1. *Спектр σ задачи* (1) *расположен на действительной положительной полуоси за исключением, быть может, конечного числа комплексно сопряжённых собственных значений конечной кратности. Существенный спектр*

σ_{ess} задачи (1) вычисляется по формуле $\sigma_{ess} = \mathcal{E} \cup \mathcal{L}$, где

$$\mathcal{E} = \{\lambda \in \mathbb{C} : \det\{2\mu_{jk} + \lambda_{jk} - \lambda^{-1}\delta_{jk}c_k\rho_{k0}(x_3)\}_{j,k=1}^2 = 0, \ \ x \in \Omega\},$$
$$\mathcal{L} = \{\lambda \in \mathbb{C} : \det\{3\mu_{jk} + \lambda_{jk} - \lambda^{-1}\delta_{jk}c_k\rho_{k0}(x_3)\}_{j,k=1}^2 = 0, \ \ x \in \partial\Omega\},$$

δ_{jk} — символ Кронекера. Множество $\sigma\backslash\sigma_{ess}$ состоит из изолированных собственных значений конечной кратности и содержит подпоследовательность с асимптотическим поведением $\lambda_k^{(\infty)} = C^{-2/3}k^{2/3}(1 + o(1))$, $k \to \infty$, где константа C может быть вычислена по матрицам вязкостей и функциям ρ_{l0} ($l = 1, 2$).

Утверждение, аналогичное теореме 1, справедливо и для смеси нескольких жидкостей. Отметим, что в работе [1] решается вопрос о существовании слабых обобщённых решений нелинейной начально-краевой задачи, описывающей баротропное движение смеси нескольких сжимаемых вязких жидкостей.

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Начальная задача для вырожденного квазилинейного уравнения с производными Герасимова–Капуто

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Пусть $\mathcal{X}$, $\mathcal{Y}$ — банаховы пространства, D_t^β — производная Герасимова–Капуто, $\mathcal{L}(\mathcal{X};\mathcal{Y})$ — множество линейных операторов, непрерывно действующих из $\mathcal{X}$ в $\mathcal{Y}$, $\mathcal{L}(\mathcal{X};\mathcal{X}) := \mathcal{L}(\mathcal{X})$, $\mathcal{C}l(\mathcal{X};\mathcal{Y})$ — множество всех линейных замкнутых операторов, плотно определенных в $\mathcal{X}$ и действующих в $\mathcal{Y}$. Пусть $L, M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$ имеют области опеделения D_L, D_M соответственно, $\ker L \neq \{0\}$. Обозначим через $\rho^L(M)$ множество таких $\mu \in \mathbb{C}$, что отображение $\mu L - M : D_L \cap D_M \to \mathcal{Y}$ инъективно, при этом $R_\mu^L(M) := (\mu L - M)^{-1}L \in \mathcal{L}(\mathcal{X})$, $L_\mu^L(M) := L(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y})$.

Пусть $m - 1 < \alpha \leqslant m \in \mathbb{N}$, $L, M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, X открыто в $\mathbb{R} \times \mathcal{X}^n$, $N : X \to \mathcal{Y}$, $T > t_0$, $f : [t_0, T] \to \mathcal{Y}$. Решением задачи

$$(Lx)^{(k)}(t_0) = y_k, \quad k = 0, 1, \dots, m - 1, \tag{1}$$

$$D_t^\alpha Lx(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots, D_t^{\alpha_n}x(t)) + f(t) \tag{2}$$

на отрезке $[t_0, t_1]$ будем называть такую функцию $x : [t_0, t_1] \to D_M \cap D_L$, что $Lx \in C^{m-1}([t_0, t_1];\mathcal{Y})$, $D_t^\alpha Lx \in C((t_0, t_1];\mathcal{Y})$, $Mx \in C((t_0, t_1];\mathcal{Y})$, выполняются условия (1), $(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots, D_t^{\alpha_n}x(t) \in X$ при всех $t \in (t_0, t_1]$ и выполняется (2).

Определение (см. [1]). Пусть $L, M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$. Пара $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$, если

(i) существуют такие $\theta_0 \in (\pi/2, \pi)$ и $a_0 \geqslant 0$, что при всех $\lambda \in S_{\theta_0,a_0}$ выполняется включение $\lambda^\alpha \in \rho^L(M)$;

(ii) при любых $\theta \in (\pi/2, \theta_0)$, $a > a_0$ существует такое $K = K(\theta, a) > 0$, что при всех $\lambda \in S_{\theta,a}$ $\max\{\|R^L_{\lambda^\alpha}(M)\|_{\mathcal{L}(\mathcal{X})}, \|L^L_{\lambda^\alpha}(M)\|_{\mathcal{L}(\mathcal{Y})}\} \leqslant \frac{K(\theta,a)}{|\lambda^{\alpha-1}(\lambda-a)|}$.

Обозначим $\ker R^L_\mu(M) := \mathcal{X}^0$, $\ker L^L_\mu(M) := \mathcal{Y}^0$. Через $\mathcal{X}^1$ ($\mathcal{Y}^1$) обозначим замыкание образа $\mathrm{im}R^L_\mu(M)$ ($\mathrm{im}L^L_\mu(M)$) в норме $\mathcal{X}$ ($\mathcal{Y}$), а через L_k (M_k) — сужение оператора L (M) на $D_{L_k} := D_L \cap \mathcal{X}^k$ ($D_{M_k} := D_M \cap \mathcal{Y}^k$), $k = 0, 1$. Определим

$$\widetilde{y}(t) := y_0 + \frac{(t-t_0)}{1!}y_1 + \dots + \frac{(t-t_0)^{m-1}}{(m-1)!}y_{m-1}, \quad \widetilde{y}_k := D_t^{\alpha_k}|_{t=t_0}\widetilde{y}(t),\ k = 1, 2, \dots, n.$$

Теорема 1. *Пусть банаховы пространства $\mathcal{X}$ и $\mathcal{Y}$ рефлексивны, выполнено $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$ или $M_1 \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n \leqslant m-1 < \alpha \leqslant m \in \mathbb{N}$, X — открытое множество в $\mathbb{R} \times \mathcal{X}^n$, $N : X \to \mathrm{im}L$, отображение $L_1^{-1}N \in C\big(X; D_{L_1^{-1}M_1}\big)$ локально липшицево по фазовым переменным, $f : [t_0, T] \to \mathcal{Y}^0 \dot{+} \mathrm{im}L$ при некотором $T > t_0$, $f \in C([t_0, T];\mathcal{Y})$, $L_1^{-1}Qf \in C([t_0, T]; D_{L_1^{-1}M_1})$, $M_0^{-1}(I-Q)f \in C^{m-1}([t_0, T];\mathcal{X})$, $y_k \in L[D_{L_1^{-1}M_1}]$ при $k = 0, 1, \dots, m-1$, $(t_0, L_1^{-1}\widetilde{y}_1 - D_t^{\alpha_1}M_0^{-1}(I-Q)f(t_0), \dots, L_1^{-1}\widetilde{y}_n - D_t^{\alpha_n}M_0^{-1}(I-Q)f(t_0)) \in X$. Тогда существует единственное решение задачи (1), (2) на отрезке $[t_0, t_1]$ при некотором $t_1 \in (t_0, T]$.*

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О математических моделях с нелинейным граничным условием

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Проведено исследование моделей деформаций упругих систем (струна, балка) с нелинейным граничным условием под воздействием внешней нагрузки. Такого рода условие возникает за счет наличия ограничителя на перемещение из состояния равновесия одного из концов рассматриваемой физической системы. В зависимости от приложенной внешней силы соответствующий конец или остается свободным, или касается граничных точек ограничителя. Отклонение изучаемой физической системы от положения равновесия описывается интегро-дифференциальным уравнением с интегралом Стилтьеса, что позволяет учитывать локализованные в отдельных точках особенности (сосредоточенные силы,

упругие опоры) и проводить поточечный анализ как решений, так и соотношений. Соответствующее уравнение является аналогом уравнения Эйлера и вместе с условиями получено вариационным методом из задачи о минимизации энергетического функционала.

Установлены необходимое и достаточное условия экстремума для рассматриваемых энергетических функционалов. Доказаны теоремы существования и единственности решений; получены формулы представления решений; исследована зависимость решений от размера ограничителя; разработан алгоритм нахождения приближенных решений.

Кроме того, исследован ряд моделей колебаний струнных систем в предположении, что ограничитель может двигаться в перпендикулярном направлении к плоскости, в которой расположена исследуемая физическая система в положении равновесия. Для таких задач доказаны теоремы существования и единственности решений, получен аналог формулы Даламбера.

Работа выполнена при финансовой поддержке Министерства науки и высшего образования РФ в рамках выполнения государственного задания в сфере науки (номер темы FZGF-2020-0009), РФФИ и НЦНИ в рамках научного проекта № 20-51-15003 НЦНИ-а.

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О некоторых важных задачах теории параболических уравнений

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В первой части исследована трехмерная задача Навье—Стокса для сжимаемого или несжимаемого потока жидкости или газа. Предложена методика исследования решения задачи с использованием классической теории параболических

уравнений, обеспечивающая существование и единственность решения уравнения Навье—Стокса в бесконечном пространстве. Для определения скалярной функции распределения давления получен закон сохранения энергии, одномерным случаем которого является уравнение Бернулли. Во второй части рассмотрена двухфазная задача Стефана в плоской клиновидной области. Построение решение задачи Стефана в рассматриваемой области основано на использовании ранее построенного автором решения задачи сопряжения в плоском угле с линией разрыва коэффициента, выходящего на нерегулярную границу области. Данный подход обеспечивает доказательство существования и единственности двумерной и двухфазной задачи Стефана и позволяет определить границу движения раздела двух сред.

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Граничные обратные задачи для сингулярных возмущений оператора Лапласа

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В работах [1–4] приведены описания корректно разрешимых краевых задач в проколотых областях для оператора Лапласа в шаре произвольного числа переменных. Указана их связь с дельтообразными возмущениями оператора Лапласа—Дирихле. Там же приведены теоремы о локализации спектров указанных краевых задач. В докладе будет обсуждаться вопрос об однозначном восстановлении области определения оператора Лапласа, возмущенного дельтообразными потенциалами. Поскольку область определения задается граничными условиями,

обсуждаемые вопросы относятся к обратным задачам. В то же время, идентификация граничных условий оператора Лапласа производится по набору спектров некоторых эталонных задач. Схема восстановления граничных условий в случае одномерных дифференциальных операторов высших порядков по набору спектров эталонных задач приведена в работе [5]. Предложенная схема восстановления граничных условий в работе [5] адаптирована для однозначного определения граничных условий оператора Лапласа с дельтообразными возмущениями.

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Бифуркация Андронова–Хопфа в логистическом уравнении с запаздыванием, диффузией и быстро осциллирующими коэффициентами

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Рассмотрено важное для приложений логистическое уравнение с запаздыванием и диффузией

$$\frac{\partial u}{\partial t} = d(\omega t)\frac{\partial^2 u}{\partial x^2} - r(\omega t)u(t - T(\omega t), x)[1 + u] \tag{1}$$

с граничными условиями

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = \gamma(\omega t)u|_{x=1}. \tag{2}$$

Положительные коэффициенты $d(s), r(s), T(s)$, а также коэффициент $\gamma(s)$ предполагаются 2π-периодическими. В качестве пространства начальных условий для (1), (2) удобно выбрать пространство $W = C[-max_s T(s), 0] \times W_2^2$. Предполагается, что все коэффициенты уравнения, а также коэффициенты в граничных условиях являются быстро осциллирующими функциями времени. Построено усредненное уравнение и рассмотрен вопрос о связи между его решениями и

решениями исходного уравнения. Особо отметим, что осцилляции коэффициента запаздывания T, например, могут приводить к появлению разнообразных классов нелинейных усредненных уравнений. Сформулирован результат об устойчивости решений и изучена задача о локальной динамике в критическом случае. Предложен алгоритм построения асимптотики решений и алгоритм исследования устойчивости. Важно отметить, что соответствующий алгоритм содержит как регулярные, так и погранслойные составляющие. Приведены содержательные примеры.

Работа выполнена при финансовой поддержке гранта Российского научного фонда, проект № 21-71-30011.

О решениях нелинейных эллиптических уравнений с L_1-данными в неограниченных областях

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В неограниченной области $\Omega \subset \mathbb{R}^n = \{\mathrm{x} = (x_1, \ldots, x_n)\}$, $n \geqslant 2$, рассматривается задача Дирихле

$$-\operatorname{div} \mathrm{a}(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) + b(\mathrm{x}, u, \nabla u) = f, \quad f \in L_1(\Omega), \quad \mathrm{x} \in \Omega, \tag{1}$$

$$u\Big|_{\partial\Omega} = 0. \tag{2}$$

Здесь функции $\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) = (a_1(\mathrm{x}, s_0, \mathrm{s}), \ldots, a_n(\mathrm{x}, s_0, \mathrm{s})) : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}^n$, $b(\mathrm{x}, s_0, \mathrm{s}) : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ имеют рост по (s_0, s), определяемый обобщенной N-функцией $M(\mathrm{x}, z)$, которая не обязана удовлетворять Δ_2-условию.

В пространствах Музилака—Орлича для уравнения

$$-\operatorname{div} \mathrm{a}(\mathrm{x}, \nabla u) = f, \quad f \in L_1(\Omega), \quad \mathrm{x} \in \Omega, \tag{3}$$

с граничным условием (2) в ограниченной области Ω в работе [1] доказано существование ренормализованного решения, а в работе [2] установлены существование и единственность энтропийного и ренормализованного решений, показана их эквивалентность.

Авторами в пространствах Музилака—Орлича в работе [3] доказано существование энтропийного решения и установлено, что оно является ренормализованным решением задачи (1), (2) в произвольной (в том числе и неограниченной) области Ω, удовлетворяющей сегментному свойству. Кроме того, для уравнения (1) с $\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) \equiv \mathrm{a}(\mathrm{x}, \mathrm{s})$, $b(\mathrm{x}, s_0, \mathrm{s}) \equiv b(\mathrm{x}, s_0)$ получены некоторые свойства и доказана единственность энтропийных и ренормализованных решений задачи Дирихле, а также установлена их эквивалентность.

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О разрешимости обобщенной задачи Неймана для эллиптического уравнения высокого порядка в бесконечной области

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В области D на плоскости, ограниченной простым гладким контуром Γ, рассмотрим эллиптическое уравнение $2l$-го порядка

$$\sum_{r=0}^{2l} a_r \frac{\partial^{2l} u}{\partial x^{2l-r} \partial y^r} + \sum_{0 \leqslant r \leqslant k \leqslant 2l-1} a_{rk}(z) \frac{\partial^k u}{\partial x^{k-r} \partial y^r} = f(z), \quad z = x + iy \in D, \tag{1}$$

с постоянными старшими коэффициентами $a_r \in \mathbb{R}$.

Исходя из набора $1 = k_1 < \ldots < k_l \leqslant 2l$ натуральных чисел, обобщенная задача Неймана для этого уравнения определяется краевыми условиями

$$\left. \frac{\partial^{k_j - 1} u}{\partial n^{k_j - 1}} \right|_{\Gamma} = f_j, \quad j = 1, \ldots, l, \tag{2}$$

где $n = n_1 + i n_2$ означает единичную внешнюю нормаль.

Постановка конкретной задачи (1), (2) при $k_{j+1} - k_j \equiv 1$ для полигармонического уравнения восходит к А. В. Бицадзе [1], где при $k_1 \geqslant 2$ она названа обобщенной задачей Неймана. Это название в дальнейшем сохраняем и для произвольного набора показателей k_j, вводя для задачи обозначение $\mathcal{N}$. Символ $\mathcal{N}_0$ сохраняем для задачи, когда все младшие коэффициенты a_{rk} в (1) равны нулю.

В конечной односвязной области D задача $\mathcal{N}$ подробно исследовалась в работах [2, 3]. В работе [3] эта задача изучалась задача в классе $C_a^{2l,\mu}(\overline{D}) = \{u \in C^{2l}(D) \cap C^{2l-1,\mu}(\overline{D}), \quad L_a u \in C^{\mu}(\overline{D})\}$.

Обозначим через ν_k, $1 \leqslant k \leqslant m$, все различные корни в верхней полуплоскости характеристического многочлена

$$\chi(z) = a_{2l} \prod_{k=1}^{m} (z - \nu_k)^{l_k} \prod_{k=1}^{m} (z - \overline{\nu}_k)^{l_k}$$

так что сумма кратностей $l_1 + \ldots + l_m$ этих корней равна l.

Введем дробно-линейные по z функции

$$\omega(e,\nu) = \frac{-e_2 + e_1\nu}{e_1 + e_2\nu},\ 1 \leqslant j \leqslant l, \tag{3}$$

где зависимость единичного касательного вектора $e = e_1 + ie_2$ к контуру Γ от нормального вектора $e = e_1 + ie_2 = i(n_1 + in_2)$. Здесь надо понимать, что $\nu = \nu_j$, $1 \leqslant j \leqslant l$.

Исходя из l-вектор-функции $g(\zeta) = (g_1(\zeta), \ldots, g_l(\zeta))$, аналитической в окрестности точек $\zeta_1, \ldots, \zeta_m$, введем блочную $l \times l-$ матрицу

$$W_g(\zeta_1, \ldots, \zeta_m) = (W_g(\zeta_1), \ldots, W_g(\zeta_m)), \tag{4}$$

где матрица $W_g(\zeta_k) \in \mathbb{C}^{l \times l_k}$ составлена из векторов-столбцов

$$g(\zeta_k), g'(\zeta_k), \ldots, \frac{1}{(l_k-1)!} g^{(l_k-1)}(\zeta_k).$$

Пусть область D бесконечна и ограничена контуром $\Gamma \in C^{2l,\nu}$, связные компоненты которого обозначим $\Gamma_0, \ldots, \Gamma_n$. Следуя [4], введем пространство Гельдера $C^{\mu}_{\lambda}(\overline{D}, \infty)$, $\lambda \in \mathbb{R}$, функций со степенным поведением $O(|z|^{\lambda})$ на бесконечности. Более точно, при $\lambda = 0$ оно состоит из ограниченных функций φ, для которых $\psi(z) = |z|^{\mu}\varphi(z)$ удовлетворяет условию Гельдера с показателем μ. Относительно нормы

$$|\varphi| = \sup_{z \in D} |\varphi(z)| + \sup_{z_1 \neq z_2} \frac{|\psi(z_1) - \psi(z_2)|}{|z_1 - z_2|^{\mu}}$$

это пространство банахово, причем оно является банаховой алгеброй по умножению. В общем случае произвольное λ банахово пространство $C^{\mu}_{\lambda}(\overline{D}, \infty)$ определим как класс функций φ, для которых $(1 + |z|)^{-\lambda}\varphi(z) \in C^{\mu}_0(\overline{D}, \infty)$, снабженный перенесенной нормой. Соответствующие банаховы пространства $C^{n,\mu}_{\lambda}(\overline{D}, \infty)$ дифференцируемых функций определим индуктивно условиями

$$\varphi \in C^n(D) \cap C^{n-1,\mu}_{\lambda}(\overline{D}, \infty), \quad \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \in C^{n-1,\mu}_{\lambda-1}(\overline{D}, \infty). \tag{5}$$

Они являются банаховыми относительно соответствующей нормы.

Поскольку в дальнейшем бесконечная область D фиксирована, пространства $C^{n,\mu}_{\lambda}(\overline{D}, \infty)$ всюду обозначаем кратко $C^{n,\mu}_{\lambda}$. В более общей ситуации конечного множества особых точек они были детально изучены в [4].

Задачу $\mathcal{N}$ рассмотрим в классе $C^{2l,\mu}_{\lambda}$, $-1 < \lambda < 0$, функций, исчезающих на бесконечности. Для нее справедлив следующий основной результат.

Теорема 1. *Пусть бесконечная область D ограничена контуром Γ класса $C^{2l,\nu}$, $\mu < \nu < 1$, состоящим из компонент $\Gamma_0, \ldots, \Gamma_n$, младшие коэффициенты уравнения* (1) *удовлетворяют требованию*

$$a_{rk} \in C^{\mu}_{k-2l-0}(\overline{D}, \infty) = \cup_{\varepsilon > 0} C^{\mu}_{k-2l-\varepsilon}, \tag{6}$$

и выполнено условие

$$\det W_g[\omega(e,\nu_1), \ldots, \omega(e,\nu_m)] \neq 0, \quad e \in \mathbb{T}, \tag{7}$$

где $\mathbb{T}$ означает единичную окружность. Тогда задача $\mathcal{N}$ фредгольмова в классе $C_\lambda^{2l,\mu}$, $-1<\lambda<0$, и ее индекс дается формулой

$$æ = 2(n+1)[æ_0 + 2\sum\nolimits_{i<j} l_i l_j] - l(2l-1). \tag{8}$$

Лемма. *Задачи $\mathcal{N}$ и $\mathcal{N}_0$ в классе $C_\lambda^{2l,\mu}$, $-1<\lambda<0$, фредгольмово эквивалентны и их индексы совпадают.*

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Об уточнении метода сплайн-коллокаций решения некоторых интегральных уравнений

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Известно, что методы, основанные на квазиинтерполяции, успешно применяются в задачах приближения решений интегральных уравнений. Например, в работах [1,2] было показано, что замена решения интегрального уравнения Фредгольма линейной комбинацией B-сплайнов, коэффициенты при которых вычисляются при помощи квазиинтерполяции функций, входящих в уравнение, позволяет получать достаточно точные приближения при использовании различных подходов к решению: метода Галёркина, метода Канторовича, метода итераций Слоана, а также более поздних обобщений.

Квазиинтерполяционные функционалы были построены для различных классов сплайнов. В работе [3] они были получены авторами доклада в общем виде и для минимальных сплайнов. Такие сплайны получаются из аппроксимационных соотношений с использованием полной цепочки векторов и порождающей вектор-функции, а также обладают минимальным носителем (см. [4]). На протяжении последних десятилетий минимальные сплайны зарекомендовали себя как хороший инструмент аппроксимации.

В данном докладе рассматривается метод сплайн-коллокаций с итерациями Слоана для интегральных уравнений Фредгольма и Вольтерры, в котором приближенное решение строится как линейная комбинация минимальных сплайнов, а в качестве коэффициентов рассматриваются значения упомянутых ранее функционалов.

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Математическое моделирование вращения расплава под воздействием импульсных нагрузок

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В докладе представлена модель распределения тока в образце вольфрама и испаряемом веществе при нагреве поверхности электронным пучком. Модель основана на решении уравнений электродинамики и двухфазной задачи Стефана для расчета температуры в области образца в цилиндрической системе координат. Ток рассматривается как возможный источник вращения вещества, который наблюдается в эксперименте. Результаты проведенного моделирования показали, что учет термотоков в парах вольфрама над пластинкой необходим для получения ускорения, способного инициировать наблюдаемое в эксперименте вращение расплава. Параметры модели взяты из экспериментов на стенде Beam of Electrons for materials Test Applications (BETA), созданного в ИЯФ СО РАН. Работа выполнена при финансовой поддержке Минобрнауки РФ в рамках государственного задания (номер темы FSSF-2020-0018).

Об оценке резольвенты одного оператора, порожденного дифференциальным уравнением второго порядка с нелокальными условиями

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Рассматривается задача

$$\begin{aligned} &u'' + \tfrac{a(x)}{x}u' - \lambda u = f, \qquad x \in (0,1), \\ &u(0) = 0, \quad \int\limits_0^1 u(x)\,dx = 0. \end{aligned} \tag{1}$$

Здесь a — непрерывная функция на отрезке $[0,1]$. Функция f принадлежит пространству $C_\alpha[0,1]$ — пространству непрерывных на интервале (0,1) функций, для которых существуют конечные пределы $\lim\limits_{x\to 0} x^\alpha f(x)$, $\lim\limits_{x\to 1}(1-x)^\alpha f(x)$. Здесь $1 < \alpha < 2$. Норма в пространстве $C_\alpha[0,1]$ задается формулой

$$\|f\|_\alpha = \sup_{0<x<1} x^\alpha (1-x)^\alpha |f(x)|.$$

Задача (1) порождает оператор $A : D(A) \subset C_\alpha[0,1] \to C_\alpha[0,1]$,

$$Au = u'' + \frac{a(x)}{x}u,'$$

с областью определения $D(A) = \{u \in C[0,1] \cap C^1(0,1) \cap C^2(0,1),\ u'' \in C_\alpha[0,1],$ $u(0) = 0,\ \int\limits_0^1 u(x)\,dx = 0\}$.

Найдены условия на функцию $a(x)$, при выполнении которых для любого $\varphi \in [0, \frac{\pi}{2})$ существует такое $r > 0$, что множество $\Sigma = \{\lambda \in C^1 : |arg\lambda| \leqslant \frac{\pi}{2} + \varphi,\ |\lambda| \geqslant r\}$ лежит в регулярном множестве оператора A и для всех $\lambda \in \Sigma$ справедлива оценка

$$\|(A - \lambda I)^{-1}\|_\alpha \leqslant \frac{C}{|\lambda|}.$$

Задача (1) при этом для всех $\lambda \in \Sigma$ имеет единственное решение, принадлежащее $D(A)$.

Анализ структур функционально-дифференциальных уравнений нелинейной оптики

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Рассматриваются модели формирования фазовых пространственных структур, в частности, для модели кольцевого резонатора, содержащего слой нелинейной среды [1]. Функционально-дифференциальные уравнения, содержащие операторы преобразования пространственных координат, рассматривались на теоретическом и экспериментальном уровнях многими коллективами (Е. П. Белан, Е. М. Варфоломеев, В. Ю. Иванов, С. А. Кащенко, А. А. Корнута, А. Н. Куликов, Д. А. Куликов, В. А. Лукьяненко, О. Б. Лыкова, А. Б. Муравник, А. В. Разгулин, Л. Е. Россовский, А. Л. Скубачевский, Ю. А. Хазова и др.).

Базовой является модель (с краевыми и начальными условиями)

$$\tau_0 u_t + u = D\Delta u + K\left|QA(x,t)\right|^2, \quad x \in \mathbb{R},$$

$$A\left(x, t+t_\tau\right) = (1-R)^{1/2} A_{in}(x) + Ke^{i\varphi_0} exp(iL\Delta)\left\{A(x,t)e^{A(x,t)e^{iu(x,t)}}\right\},$$

$$-2ik_0\frac{\partial A(x,z,t)}{\partial z} = \Delta A(x,z,t), \quad A(x,t) = A(x,z=0,t),$$

где τ_0, D, K, R, φ_0, L, k_0 — параметры оптической системы; $A_{in}(x)$ — амплитуда входной световой волны; Q — оператор преобразования пространственных переменных. Для стационарных решений $u = u_s$, $A = A_s$, $A_{in} = A_{ins}$ имеют место соотношения
$u_s = K\left|A_s\right|^2 = (1-R)k\left[1 - 2Rcos\left(u_s + \varphi_0\right) + R^2\right]^{-1}$, $k = KI_0$, $I_0 = \left|A_{ins}\right|^2$, $A_s = \left(1 - Re^{i(u_s+\varphi_0)}\right)^{-1}\sqrt{1-R}A_{ins}$.

Для линеаризованной системы в окрестности (u_s, A_s, A_{ins}) найдены собственные значения и собственные функции, получены условия устойчивости, начально-краевая задача представлена в виде интегральных уравнений. Для частных случаев построены асимптотические представления для решений.

Данная работа продолжает исследования, представленные в работах [2–4].

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О существовании и единственности ренормализованного решения для эллиптического уравнения с мерозначным потенциалом

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На компактном многообразии $\mathcal{M}$ с краем рассматривается задача

$$-\mathrm{div}_g(a(x,u,du)) + \Lambda u + Ku = f, \quad u|_{\partial\mathcal{M}} = 0; \quad f \in L_1(\mathcal{M}). \tag{1}$$

Оператор $Ku : \mathcal{H}^1_{p(\cdot)}(\mathcal{M}) \to (\mathcal{H}^1_{p(\cdot)}(\mathcal{M}))^*$ компактен, $\Lambda u = -\mathrm{div}_g(F\nabla_g u)$, где векторное поле $F\nabla_g u$ в локальной системе координат имеет выражение $F_i^k g^{ij} u_{x^j}$. Функции $F_i^k(x)$ принадлежат $L_{1,\mathrm{loc}}(\mathcal{M})$, и при любом $x \in \mathcal{M}$ элементы $F_i^k(x)$ задают неотрицательно определенную матрицу.

Модельным примером оператора K является оператор $K(u) = \mu|u|^{q-1}u$, где μ — мера Радона.

Векторное поле a при $r \in \mathbb{R}$ удовлетворяет условиям ограниченности, монотонности при $y \neq z$ $(a(x,r,y) - a(x,r,z), y - z) > 0$, коэрцитивности

$$(a(x,r,y),y) \geqslant 2\delta_0|y|^{p(x)} - G(x), \quad \text{где} \quad G(x) \in L_1(\mathcal{M}), \quad y,z \in T^*_x\mathcal{M}, \quad x \in \mathcal{M}.$$

Пространство $\mathbb{B}_{p(\cdot)}(\mathcal{M})$ — пополнение $\mathcal{D}(\mathcal{M})$ по норме

$$\|u\|_{p(\cdot),\Lambda} = \|u\|_{p(\cdot),1} + \sqrt{\langle \Lambda u, u\rangle}.$$

Положим $T_k(r) = \max(-k, \min(r,k))$.

Определение 1. Измеримая функция u называется ренормализованным решением задачи Дирихле (1), если она удовлетворяет соотношениям: $T_k(u) \in \mathbb{B}_{p(\cdot)}(\mathcal{M})$ при всех $k > 0$;

$$\lim_{k\to\infty} \int\limits_{\mathcal{M}, k\leqslant|u|\leqslant k+1} (a(x,u,du),du)d\nu = 0;$$

при всех $\xi \in \mathrm{Lip}_0(\mathbb{R})$, $v \in \mathcal{D}(\mathcal{M})$ выполнено равенство

$$\int\limits_{\mathcal{M}} (a(x,u,du), d(v\xi(u)))d\nu + \langle \Lambda u + Ku, v\xi(u)\rangle = \langle f, v\xi(u)\rangle. \tag{2}$$

Пусть существуют такое число $A > 1$, что

$$\langle \Lambda u + Ku, u\rangle + \delta_0 \int\limits_{\mathcal{M}} |\nabla_g u|_g^{p(x)} d\nu \geqslant 0, \qquad u \in \mathbb{B}_{p(\cdot)}(\mathcal{M}), \tag{3}$$

при $\|u\|_{p(\cdot),1} \geqslant A$.

Теорема 1. *Пусть выполнены условия на a, F, K и (3). Тогда существует ренормализованное решение задачи (1).*

При некоторых дополнительных условиях установлена единственность ренормализованного решения.

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Эффективные квазиклассические асимптотики

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Канонический оператор, созданный В. П. Масловым [1] в 1965 г., является одним из мощнейших средств построения глобальных квазиклассических асимптотик. Он основан на геометрическом объекте — лагранжевом многообразии Λ в фазовом пространстве $\mathbb{R}^{2n}_{(x,p)}$. Канонический оператор действует на гладкие функции на Λ и переводит их в быстроосциллирующие функции на конфигурационном пространстве $\mathbb{R}^n_x$, зависящие от малого параметра $h > 0$. В практических расчетах важно, чтобы асимптотика была эффективной, т. е. позволяла исследовать задачу достаточно быстро и с достаточно скромными вычислительными затратами. Понятие эффективности зависит от имеющихся вычислительных средств и существенно изменилось с появлением систем технических вычислений *Wolfram Mathematica*, *MatLab* и им подобных, предоставляющих принципиально новые возможности оперативной реализации и визуализации вычислений. Классическое определение канонического оператора не всегда удовлетворяет требованиям эффективности, отчасти потому, что используемые им локальные координаты на лагранжевом многообразии априори никак не связаны с решаемой задачей. В докладе дан обзор современных версий и модификаций канонического оператора, позволяющих строить эффективные асимптотики. К соответствующим конструкциям относятся:

- новые формулы, позволяющие записать канонический оператор в произвольных координатах на лагранжевом многообразии;
- представления канонического оператора в окрестности общих каустик в виде специальных функций (Эйри, Пирси и др.) сложного аргумента;
- в задачах с локализованными начальными данными — обобщения канонического оператора, позволяющие исследовать случаи, когда эффективные гамильтонианы (и, как следствие, лагранжевы многообразия) имеют особенность специального вида при $p = 0$ (сюда относятся, например, волновое уравнение, системы, гиперболические по Петровскому, псевдодифференциальное уравнение, описывающее волны на воде в линеаризованном приближении с учетом дисперсии и т. д.).

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Нелинейные волны и солитоны

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Будет дан обзор периодических проблем нелинейных волн и солитонов и их связь с квантовой механикой.

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О неустойчивых состояниях равновесия двумерной системы Броудвелла

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Кинетическая теория рассматривает газ как совокупность громадного числа хаотически движущихся частиц тем или иным образом взаимодействующих между собой [2, 3]. В результате таких взаимодействий частицы обмениваются импульсами, энергией. Взаимодействие может осуществляться путем прямого столкновения частиц или при помощи тех или иных сил.

Для пояснения математической схемы, описывающей подобные явления, в [2] рассматриваются так называемые дискретные модели кинетического уравнения Больцмана и приводится феноменологический вывод уравнения Больцмана для газовой модели с конечным числом различных скоростей частиц и конечным числом разных взаимодействий (модели типа Броудвелла [1]).

Двумерная модель Броудвелла

$$\begin{aligned}
\partial_t n_1 + c\partial_x n_1 &= \frac{1}{\varepsilon}(n_3 n_4 - n_1 n_2),\\
\partial_t n_2 + c\partial_x n_2 &= \frac{1}{\varepsilon}(n_3 n_4 - n_1 n_2),\\
\partial_t n_3 + c\partial_y n_3 &= \frac{1}{\varepsilon}(n_1 n_2 - n_3 n_4),\\
\partial_t n_4 + c\partial_y n_4 &= \frac{1}{\varepsilon}(n_1 n_2 - n_3 n_4),
\end{aligned} \tag{1}$$

(см. там же [4]) относится к классу неинтегрируемых уравнений. Система (1) является кинетическим уравнением Больцмана модельного двумерного газа [1] частиц движущихся на двумерной плоскости, скорости которых $(c, 0)$, $(-c, 0)$, $(0, c)$, $(0, -c)$ будем предполагать направленными вдоль координатных осей. Здесь $n_1(x, y, t)$, $n_2(x, y, t)$, $n_3(x, y, t)$, $n_4(x, y, t)$ — плотность (число частиц на единицу площади) частиц соответствующих четырех групп. Все частицы распределены по четырем группам со скоростями $(c, 0)$, $(-c, 0)$, $(0, c)$, $(0, -c)$, обмениваются скоростями $-I + II = II + I$, переходя в частицы третьей и четвертой групп $(I + II = III + IV)$. Аналогично $III + IV = IV + III$ или $III + IV = I + II$, $I + III = III + I$, $I + IV = IV + I$, $II + III = III + II$, $II + IV = IV + II$.

Изменение числа частиц в группах может происходить только в результате реакций:

$$I + II = III + IV, \quad III + IV = I + II.$$

Как показано в [2], $n_i > 0$, $i = 1, \ldots, 4$, если начальные условия $n_i^0 > 0$, $i = 1, \ldots, 4$. Здесь выполнены уравнение неразрывности и уравнения сохранения импульса.

Эта система обнаруживает нерегулярное поведение решений (численным экспериментом устанавливает неустойчивость стационарных решений при некоторых значениях внутренних параметров). Наша задача — установить это аналитически.

Теорема. *Рассмотрим малые периодические возмущения состояния равновесия*

$$\begin{aligned} n_1 &= n_1^e + \varepsilon^2\sqrt{n_1^e}\hat{u}, \quad n_2 = n_2^e + \varepsilon^2\sqrt{n_2^e}\hat{v}, \\ n_3 &= n_3^e + \varepsilon^2\sqrt{n_3^e}\hat{w}, \quad n_4 = n_4^e + \varepsilon^2\sqrt{n_4^e}\hat{z}. \end{aligned} \tag{2}$$

Докажем, что положительные состояния равновесия $(n_1^e, n_2^e, n_3^e, n_4^e)$, $n_j^e > 0$, $n_1^e n_2^e = n_3^e n_4^e$ *при условии* $n_3^e > n_4^e$, $n_4^e > n_1^e$ *являются седлами. Устойчивое многообразие определяется соотношениями «креста»*

$$\left(u_{k,-k}^0 + \frac{z_e^{1/2}}{u_e^{1/2} z_{k,-k}^0}\right) = \left(v_{k,k}^0 + \frac{z_e^{1/2}}{v_e^{1/2} z_{k,k}^0}\right), \quad k \in \mathbb{Z},$$

на коэффициенты Фурье начального возмущения (u^0, v^0, w^0, z^0). *При достаточно гладких начальных данных стабилизация вдоль устойчивого многообразия экспоненциальная, т.е. существует* $\gamma = O(\varepsilon) > 0$, *такое что*

$$\sup_Q (|u| + |v| + |w| + |z|)(t) \leqslant Ce^{-\gamma t},$$

где Q *— ячейка периодичности.*

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Операторы преобразования: некоторые современные результаты

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В докладе будут изложены некоторые современные исследования по теории операторов преобразования и их приложениям к дифференциальным уравнениям. Рассматриваются определённые типы дифференциальных уравнений с особенностями в коэффициентах, основное внимание уделяется дифференциальным уравнениям с операторами Бесселя

$$B_\nu u(x) = \frac{d^2}{dx^2}u(x) + \frac{\nu}{x}\frac{d}{dx}u(x).$$

В докладе современные исследования по теории операторов преобразования и их приложениям к дифференциальным уравнениям рассматриваются на основе ряда последних публикаций по этой тематике, см. [1–5]. Таким образом, теория операторов преобразования и их многочисленных приложений является живой и активной ветвью современной математики. Операторам преобразования и их различным применениям посвящено достаточное число публикаций, в том числе издающихся монографий и сборников.

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Коллокационно-вариационный подход для решения дифференциально-алгебраических уравнений

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В докладе рассмотрена задача

$$A(t)x^{'}(t) + B(t)x(t) = f(t),\ x(0) = x_0,\ t \in [0,1],$$

где $A(t)$, $B(t)$ — $(n \times n)$-матрицы, $f(t)$ и $x(t)$ — заданная и искомая n-мерные вектор-функции, элементы матриц $A(t)$, $B(t)$ и $f(t)$ достаточно гладкие, и

$$detA \equiv 0.$$

Такие задачи принято называть дифференциально-алгебраическими уравнениями (ДАУ). Предполагается, что начальное условие задано корректно, а из исходной задачи и r ее производных путем элементарных преобразований можно выделить классическую систему обыкновенных дифференциальных уравнений $x^{'}(t) + \bar{B}(t)x(t) = \bar{f}(t)$. Значение r принято называть индексом рассматриваемой задачи. Многие известные неявные методы могут порождать неустойчивый процесс или принципиально неприменимы для ДАУ.

Авторы для таких задач предлагают коллокационно-вариационные разностные схемы, которые имеют принципиальное отличие от классических алгоритмов. Описан общий подход к созданию коллокационно-вариационных разностных схем, основанный на построении задачи математического программирования специального вида. Приведены конкретные алгоритмы с одной и двумя точками коллокации и результаты численных расчетов. Данное исследование продолжает работы [1, 2].

Работа выполнена при поддержке Российского фонда фундаментальных исследований (проект 20-51-S52003).

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Линеаризация с помощью функционального параметра

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Рассмотрим гамильтонову систему около положения равновесия или симплектическое отображение около неподвижной точки в фазовом пространстве размерности $2n$. Предположим, что система зависит от функционального параметра, являющегося функцией n переменных. Изучается возможность использовать функциональный параметр для получения системы, сопряженной линейной на открытом множестве.

Квазилинейные уравнения с несколькими производными Римана—Лиувилля в секториальном случае

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Пусть D_t^β — дробная производная Римана—Лиувилля, $\mathcal{Z}$ — банахово пространство, Z — открытое множество в $\mathbb{R}\times\mathcal{Z}^m$, $F: Z\to\mathcal{Z}$. Рассмотрим квазилинейное уравнение

$$D_t^\alpha z(t)=\sum_{j=1}^{m-1}A_jD_t^{\alpha-m+j}z(t)+\sum_{l=1}^{n}B_lD_t^{\alpha_l}z(t)+\sum_{s=1}^{r}C_sJ_t^{\beta_s}z(t)+$$

$$+F(t,D_t^{\alpha-m}z(t),D_t^{\alpha-m+1}z(t),\dots,D_t^{\alpha-1}z(t),\quad t\in(0,T),\tag{1}$$

где $m-1<\alpha\leqslant m\in\mathbb{N}$, $0<\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leqslant m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $l=1,2,\dots,n$, $\beta_1>\beta_2>\dots>\beta_r\geqslant 0$, $A_j\in\mathcal{C}l(\mathcal{Z})$, $j=1,2,\dots,m-1$, $B_l\in\mathcal{C}l(\mathcal{Z})$, $l=1,2,\dots,n$, $C_s\in\mathcal{C}l(\mathcal{Z})$, $s=1,2,\dots,r$, т. е. мы имеем замкнутые линейные операторы с областями определения $D_{A_1},D_{A_2},\dots,D_{A_{m-1}}$, $D_{B_1},D_{B_2},\dots,D_{B_n}$, $D_{C_1},D_{C_2},\dots,D_{C_r}$ соответственно, пересечение которых плотно в $\mathcal{Z}$ и снабжено суммой норм графика этих операторов. Решением неполной задачи типа Коши

$$D_t^{\alpha-m+k}z(t_0)=z_k,\ k=m^*,m^*+1,\dots,m-1,\tag{2}$$

для уравнения (1) на $(t_0,t_1]$ назовём такую функцию $z:(t_0,t_1]\to\mathcal{Z}$, что $J_t^{m-\alpha}z\in C^m((t_0,t_1];\mathcal{Z})\cap C^{m-1}([t_0,t_1];\mathcal{Z})$, $J_t^{m-\alpha}z\in C^j((t_0,t_1];D_{A_j})$, $j=1,2,\dots,m-1$, $J_t^{m_l-\alpha_l}z\in C^{m_l}((t_0,t_1];D_{B_l})$, $l=1,2,\dots,n$, $J_t^{\beta_s}z\in C((t_0,t_1];D_{C_s})$, $s=1,2,\dots,r$, и для всех $t\in(t_0,t_1]$ выполняется вложение $(t,D_t^{\alpha-m}z(t),\dots,D_t^{\alpha-1}z(t))\in Z$, верно равенство (1) и выполняются условия (2). Здесь m^* — дефект задачи типа Коши [1], который определяется набором чисел $\alpha,\alpha_1,\alpha_2,\dots,\alpha_n$.

Теорема. *Пусть* $m-1<\alpha\leqslant m\in\mathbb{N}$, $A_j\in\mathcal{C}l(\mathcal{Z})$, $j=1,2,\ldots,m-1$, $0<\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leqslant m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $B_l\in\mathcal{C}l(\mathcal{Z})$, $l=1,2,\ldots,n$, $\beta_1>\beta_2>\cdots>\beta_r\geqslant 0$, $C_s\in\mathcal{C}l(\mathcal{Z})$, $s=1,2,\ldots,r$, $(A_1,A_2,\ldots,A_{m-1},B_1,B_2,\ldots,B_n,C_1,C_2,\ldots,C_r)\in\mathcal{A}_\alpha^{n,r}(\theta_0,a_0)$, $z_k\in D$, $k=m^*$, $m^*+1,\ldots,m-1$, Z — *открытое множество в* $\mathbb{R}\times\mathcal{Z}^m$, $(t_0,\ 0,\ 0,\ \ldots,\ 0,\ z_{m^*},\ z_{m^*+1},\ \ldots,\ z_{m-1})\in Z$, *отображение* $F\in C(Z;D)$ *локально липшицево по фазовым переменным. Тогда существует такое* $t_1>t_0$, *что задача* (1), (2) *имеет единственное решение на* $(t_0,t_1]$.

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Задача Коши для одного класса уравнений с распределенной дробной производной Герасимова—Капуто

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Пусть $\mathcal{Z}$ — банахово пространство, $\mathcal{C}l(\mathcal{Z})$ — множество всех линейных замкнутых операторов, плотно определенных в $\mathcal{Z}$, действующих в пространство $\mathcal{Z}$, D_t^α — дробная производная Герасимова—Капуто. Для $\omega\in L_1(0,b)$ обозначим $W(\lambda):=\int\limits_0^b\omega(\alpha)\lambda^\alpha d\alpha$, $\overline{\mathbb{R}}_+:=\mathbb{R}_+\cup\{0\}$.

Пусть $K\geqslant 1$, $a\geqslant 0$, и оператор $A\in\mathcal{C}l(\mathcal{Z})$ удовлетворяет следующим условиям:

1. при $\Re\lambda>a$ выполняется включение $W(\lambda)\in\rho(A)$;
2. существует такое $K>0$, что при всех $\Re\lambda>a$, $n\in\mathbb{N}\cup\{0\}$ выполняется неравенство

$$\left\|\frac{d^n}{d\lambda^n}\left(\frac{W(\lambda)}{\lambda}(W(\lambda)I-A)^{-1}\right)\right\|_{\mathcal{L}(\mathcal{Z})}\leqslant\frac{Kn!}{(\Re\lambda-a)^{n+1}}.$$

В таком случае мы будем говорить, что оператор A принадлежит классу $\mathcal{C}_W(K,a)$ [1]. Через D_A будем обозначать область определения оператора A, снабженную нормой его графика.

Теорема. *Пусть* $b \in (1,2]$, $\omega \in L_1(0,b)$, $A \in \mathcal{C}_W(K,a)$ *при некоторых* $K \geqslant 1$, $a \geqslant 0$; $z_0, z_1 \in D_A$. *Тогда существует единственное решение задачи Коши* $z(0) = z_0, z^{(1)}(0) = z_1$ *для уравнения с распределенной дробной производной Герасимова—Капуто*

$$\int\limits_0^b \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t), \quad t > 0.$$

Под решением задачи Коши понимается функция $z \in C(\mathbb{R}_+; D_A) \cap C^1(\overline{\mathbb{R}}_+; D_A)$, для которой $\int\limits_0^b \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbb{R}_+; \mathcal{Z})$.

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Спектр скоростей потоков на открытых и замкнутых цепочках как непрерывных или дискретных динамических системах

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В центре внимания эргодической теории находится изучение асимптотического поведения динамических систем на многообразиях. При описании потоков на многообразиях плодотворным является метод аппроксимации автоморфизмов периодическими преобразованиями, см. [1].

Модели автомобильного трафика как движения частиц в одном направлении сводятся к вероятностным методам случайных процессов с запретами [2], аппроксимациями которых являются динамические системы, порожденные итерациями отображений на многообразии.

Мы исследуем динамические системы типа сетей Буслаева [3], с непрерывным пространством состояний и непрерывным временем, а также с дискретным пространством состоянии и дискретным временем. Носителем контурной сети является система контуров с общими узлами, определяющими конфликты и задержки, см. [4]. В дискретном варианте контур разбит на ячейки, в которых располагаются частицы, перемещающиеся по заданным правилам. В непрерывном случае частицы одного кластера перемещаются одновременно, см. [5].

Инвариантные множества состояний системы являются предельными циклами, т.е. реализуются замкнутые траектории в пространстве состояний. Исследовано поведение систем на предельных циклах, что позволяет найти инвариантную меру на пространстве состояний и среднюю скорость частиц с учетом задержек, которая представляет собой важнейшую характеристику.

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